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BASIC ELECTRICAL ENGINEERING

for Students of Electrical Engineering

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NEW YORK
JOHN WILEY & SONS, INC.
CHAPMAN & HALL, LTD., LONDON
1949

PRINTED IN THE UNITED ST.TES OF AMERICA

Preface

The ever-increasing scope of electrical engineering courses imposes additional requirements on the basic courses. This textbook has been written with the immediate aim of preparing electrical engineering students for courses in alternating-current circuits, engineering electronics, and electrical machinery.

Since the amount of time devoted to first courses varies to some extent in the different schools, the most essential features of a first course are contained in the first ten chapters of the text. It is hoped that the sequence of topics is such that an orderly three-, four-, or five-semester-hour course is presented.

The text material is written at a mathematical level which is consistent with concurrent registration in calculus. Emphasis is placed upon the physical significance of the few integral forms which are used to define properly electrical potential, time-averaged currents and voltages, and space-averaged flux densities. Throughout the treatment of electric and magnetic fields, emphasis is placed upon the vector properties of electric field intensity and magnetic flux density. Field theory generally is so framed that in later courses it can be readily extended to include Laplace's, Poisson's, and Maxwell's equations.

Magnetic field theory, the enigma of first courses generally, is based upon the magnetic properties of a current-carrying loop. The magnetic properties of a current loop are first established by demonstrating the equivalence of a current loop and a magnetic pole. After the magnetic properties of a current loop are established, magnetically developed forces and magnetically generated voltages follow as natural consequences of the dimensional properties of the magnetic field.

Rationalized mks units have been used almost exclusively in this text since field theory cloaked in this system of units appears much simpler and more straightforward to beginning students than it does in other systems of units. Moreover, the rationalized mks system of units is steadily gaining prominence in the literature and may eventually become the accepted system of units in electrical engineering literature. One of the advantages of the rationalized mks system of units is that the *ampere-turn* is the primary unit of magnetomotive force, and this fact alone justifies the use of the system in a basic course.

Secondary units like *ampere-turns per inch* and *ohms per circular-mil foot*, which are widely used in engineering practice, have, of course, been employed at appropriate places throughout the text. Wherever secondary units of this kind are encountered, however, the methods of conversion together with the actual conversion factors are given.

It is a pleasure to acknowledge the invaluable assistance and criticism which I have received from my associates, particularly from Dr. H. R. Reed, Professor T. T. Witkowski, Mr. Henry W. Price, Jr., and Mr. Walter R. Beam, Jr.

G. F. C.

College Park, Maryland
January 2, 1949

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CHAPTER I

Classification of Physical Quantities

"I often say that when you can measure what you are speaking about, and express it in numbers, you know something about it; but when you cannot measure it, when you cannot express it in numbers, your knowledge is of a meagre and unsatisfactory kind; it may be the beginning of knowledge, but you have scarcely, in your thoughts, advanced to the stage of science, whatever the matter may be."

LORD KELVIN, 1883

The purpose of this chapter is to review some of the physical quantities with which the reader is already familiar. A few of the more basic mechanical quantities will be considered both from a conceptual point of view and from an algebraic point of view; not with the intent of reviewing the subject of mechanics but with the intent of showing how physical quantities are classified and defined.

1. Defined and Undefined Quantities. In the study of electricity and magnetism, a wide variety of physical quantities is encountered. An orderly classification of these physical quantities requires that we select a *minimum number of fundamental quantities* to be used in defining all other physical quantities. Just which quantities are taken as *fundamental* is not so important as how many (or how few) are taken.

If elementary mechanical concepts only are involved, three fundamental physical quantities are sufficient for the purposes of defining all other mechanical quantities. If thermal concepts are involved, we must add a fourth quantity to our list of fundamental (or undefined) quantities; and if electrical concepts are involved, we must add still another physical quantity to this list.

The fundamental physical quantities usually selected are

length, time, mass, temperature, electrical charge

Although these quantities may be described, they cannot be defined in the same sense that all other physical quantities are defined. This distinction between *fundamental* or *undefined* quantities and *defined* quantities follows as a natural consequence of the limitations of language. When something is defined, it is necessarily defined in terms of some-

thing else, so ultimately there is a group of quantities that remains undefined. Some measure of confusion may be avoided if, at the outset, the five physical quantities listed above are accepted as fundamental quantities which can be discussed and described possibly at great length but *not defined.*

FUNDAMENTAL MECHANICAL QUANTITIES

2. Length or Displacement. Although we cannot even describe one-dimensional space in terms of something else, we know intuitively what is meant by *length* or *displacement.* We know, moreover, that where a specified length is stated simply as a number of units the direction along

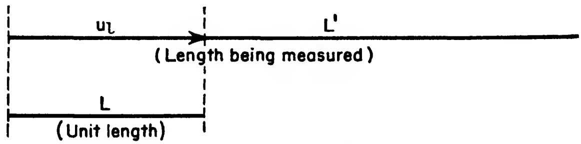

FIG. 1. Length.

which this length (or displacement) is to be taken is tacitly or otherwise implied; in other words, we know that *length* considered as a physical quantity must, in general, be treated as a *vector* quantity.

In a particular case, as for example in Fig. 1, we say algebraically that

$$l \text{ (length)} = \frac{L' \text{ (magnitude being measured)}}{L \text{ (magnitude of unit measure)}} = N\boldsymbol{u}_l \qquad (1)$$

where N is a pure number (equal to L'/L) and $\boldsymbol{u}_l$ is unit length which, to be complete, must somehow convey to the reader the direction along which L' is being reckoned. In many cases, as for example in a roll of wire 500 ft in length, we are not concerned with the direction of $\boldsymbol{u}_l$.

It should be noted that equation (1) tells us nothing about the physical quantity we call length aside from telling us how to specify the magnitude of L' in terms of an accepted unit of length; something we undoubtedly knew at the outset.

3. Time. Time or duration is a second fundamental physical quantity which is difficult to describe but which can be measured in terms of some arbitrarily selected unit, say T. Algebraically,

$$t \text{ (time)} = \frac{T' \text{ (magnitude of duration between two events)}}{T \text{ (magnitude of duration between two events which mark the beginning and end of unit time measure)}} = N\boldsymbol{u}_T \qquad (2)$$

where N is the ratio T'/T and u_T specifies the units in which T' is expressed.

Thus we express time as a ratio without attempting in any way to describe the physical quantity itself. Where ordinary or non-relativistic concepts are involved we know intuitively what is meant by time, and because units of time are comparatively easy to reproduce we employ time t as a fundamental physical quantity. *Time* is a *scalar* quantity since it possesses no directional properties.

4. Mass. Another fundamental physical quantity (in addition to length and time) is required in Newtonian mechanics because many physical quantities involving matter cannot be defined in terms of length and time alone. Neither the density of a stationary body nor the momentum of a moving body, for example, can be adequately defined in terms of length and time. The reader is undoubtedly familiar with the fact that the *mass* of a body is ordinarily specified in terms of the earth's gravitational pull upon the body. The greater this pull (or weight) the greater the mass of the body. More specifically,

$$m \text{ (mass)} = \underset{\text{(in a gravitational system)}}{\frac{\text{weight of body acted upon}}{\text{acceleration due to gravity}}} = \underset{\text{(in a general inertial system)}}{\frac{\text{unbalanced force acting}}{\text{acceleration produced}}}$$

$$m = \frac{w}{g} = \frac{f}{a} \tag{3}$$

In the above equation, weight w is the force of attraction between the body, the mass of which is being specified, and the earth. As such the numerical value of w which is a *force* is dependent upon the position of the body with respect to the center of mass of the earth. Since mass is to be employed as an invariant quantity, the ratio w/g must remain invariant.

In a general system of mechanics, $m = w/g$ is simply a special case of $m = f/a$, and any system of mechanics which attempts to eliminate the concept of mass by employing w/g for mass is altogether too restricted for our purposes. Electrically charged particles, for example, may be acted upon by forces of either attraction or repulsion which are so great as to make ordinary gravitational forces negligibly small. Our purposes will best be served by considering that *mass* is an invariant fundamental property of matter.[1]

[1] The understanding here is that the chemical structure of the matter remains unchanged and that velocities less than about 20 per cent the velocity of light are involved.

DERIVED MECHANICAL QUANTITIES

5. Defining Equations and Dimensional Forms. Throughout the study of electricity and magnetism, various physical quantities are defined in terms of the four fundamental quantities:

length, time, mass, electric charge

One of the prime objectives of a "fundamentals" course is that of acquainting the reader with the interrelationships which exist between the various electric and magnetic quantities. The general method of attack is to learn the defining equation for the new physical quantity in terms of other more familiar quantities and then proceed to examine the *dimensions* of the new quantity to see wherein the new quantity differs or is similar to other physical quantities which have physical significance to the reader.

Dimensional quality or simply the word *dimension*, as used here, refers to the classification or category of the physical quantity and does not

TABLE 1

Derived Physical Quantity	Symbol	Customary Defining Equation	Fundamental Dimensions
1. Velocity	v	$v = \frac{l}{t}$	$v = l^1t^{-1}$
2. Acceleration	a	$a = \frac{l}{t^2}$	$a = l^1t^{-2}$
3. Area	A	$A = l_1l_2$	$A = l^2$
4. Force	f	$f = ma$	$f = m^1l^1t^{-2}$
5. Gravitational acceleration	g	$g = \frac{w}{m} = \frac{f}{m}$	$g = l^1t^{-2}$
6. Momentum	M	$M = mv$	$M = m^1l^1t^{-1}$
7. Energy (mechanical)	W	$W = fl\cos\theta\Big]_f^l$	$W = m^1l^2t^{-2}$
8. Potential energy	W_p	$W_p = mgl_1 = mgh$	$W_p = m^1l^2t^{-2}$
9. Kinetic energy	W_k	$W_k = \frac{1}{2}mv^2$	$W_k = m^1l^2t^{-2}$
10. Power	P	$P = \frac{W}{t}$	$P = m^1l^2t^{-3}$

refer to the numerical magnitude of a measurement of that quantity. A study of the right-hand column of Table I will illustrate what is meant by the *dimensions* of a derived quantity in terms of the fundamental quantities mass, length, and time. The presence of the exponents in the right-hand member of a dimensional equation distinguishes this type of equation from an ordinary algebraic equation. Since a dimensional equation is not employed in obtaining numerical values, no numerical coefficients are included in this type of equation. In line 9 of Table I,

for example, the numerical factor ½ which appears in the defining equation of kinetic energy is not carried into the *dimensions* column since this factor is a numeric and as such is dimensionless.

In line 5 of Table I, w refers to the weight (or force) with which the earth attracts the body of mass m; and, in line 8, $l_1 = h$ refers to the height (or displacement) that mass m is above the zero potential energy datum plane which in a particular instance might be taken as the earth's surface.

The dimensions of a physical quantity when reduced to fundamental quantities are often helpful in appreciating the similarity of, or the difference between, physical quantities. Consider, for example, the three defining equations for W, W_p, and W_k in lines 7, 8, and 9 of Table I. Whereas the three defining equations appear to have distinctly different forms, the dimensions of the three quantities are identical as, of course, they must be if they are correctly named.

Dimensional equations are used chiefly in this text in the interpretation of the defining equations which are given for electric and magnetic quantities. In general, the defining equations are similar to those given in Table I. Dimensional interpretations of these defining equations provide a direct and logical method of learning precisely what is meant by electric potential, electric flux, and the like.

Example. Let it be required to show that W, W_p, and W_k in lines 7, 8, and 9 of Table I are all dimensionally equivalent to force times length, f^1l^1.

(*a*) Since $\cos\theta$ in line 7 is the ratio of two lengths, it is dimensionless, and hence

$$W = f^1l^1$$

(*b*) Since mg in line 8 is dimensionally the same as $ma = f$, it is plain that

$$W_p = m^1a^1l^1 = f^1l^1$$

(*c*) Since $W_k = m^1l^2t^{-2}$ and since $f = m^1l^1t^{-2}$

$$W_k = f^1l^1$$

This example illustrates how dimensional equality may be established in terms of a well-known derived physical quantity like force.

UNITS

6. Primary Units. If only mechanical quantities are involved, the matter of units may be settled simply and straightforwardly. One suitable unit of each of three fundamental quantities is first selected. The three units selected may be the centimeter, gram (mass), and second; the foot, pound (force), and second; or any other three units which have

TABLE II

System of Units	cgs	mks	(American) fps
Units of Fundamental Quantities	unit length, cm unit mass, g unit time, sec	unit length, m unit mass, kg unit time, sec	unit length, ft unit force, lb unit time, sec
Ratios of Like Fundamental Units in Different Systems	length ratios	$\frac{\text{cm}}{\text{m}} = 100$	$\frac{\text{cm}}{\text{ft}} = 30.48$
	mass ratios	$\frac{\text{g}}{\text{kg}} = 1000$ $1 \text{ kg} \equiv 1000 \text{ g}$	$\frac{\text{g}}{\text{slugs}} = 14{,}600$* $1 \text{ slug} \equiv 14{,}600 \text{ g}$

Physical Quantity	Symbol	Defining Equation		Derived Units	
Area	A	$A = l_1 l_2$	sq cm	sq m	sq ft
Volume	V	$V = l_1 l_2 l_3$	cu cm	cu m	cu ft
Mass density	D	D = mass/volume	g/cu cm	kg/cu m	slugs/cu ft
Velocity	v	$v = l/t$	cm/sec	m/sec	ft/sec
Acceleration	a	$a = v/t$	cm/sec/sec	m/sec/sec	ft/sec/sec
Momentum	M	M = mass × velocity	$\text{g} \times \frac{\text{cm}}{\text{sec}}$	$\text{kg} \times \frac{\text{m}}{\text{sec}}$	$\text{slugs} \times \frac{\text{ft}}{\text{sec}}$
Force	f	f = mass × acceleration	dynes	newtons (1 newton $\equiv 10^5$ dynes)	pounds (1 lb $\equiv$ 445,000 dynes)

Unit force imparts unit acceleration to a mass of:			1 g	1 kg or to a body which weighs 2 205 lb†	1 slug or to a body which weighs 32 19 lb‡
Weight of unit mass	w	$w = mg$	981 dynes	9 81 newtons	32 19 lb
Mechanical work or energy	W	$W = fl$	ergs (dyne-cm)	newton-meters (or joules)	ft-lb
Power	p	$p = \frac{\text{work}}{\text{time}} = \frac{W}{t}$	ergs/sec (or abwatts)	joules/sec (or watts)	ft-lb/sec
Temperature	T	(basic or fundamental)	°C	°C	°F
Unit of heat energy	Q_h	heat required to raise unit mass of HOH 1°	g-cal (1 g HOH 1°C)	kg-cal (1 kg of HOH 1°C)	Btu (1 lb weight HOH 1°F)
Mechanical equivalent of heat	J_h	$\frac{\text{mech energy}}{\text{heat energy}}$ (as a ratio) (empirically determined)	$\frac{\text{ergs}}{\text{g-cal}} = 4\,186 \times 10^7$	$\frac{\text{joules}}{\text{kg-cal}} = 4186$	$\frac{\text{ft-lb}}{\text{Btu}} = 778$

Note 1 In rounding to 981 cm sec/sec we take the weight of a 453 6-g mass as 981 × 453 6 or 445 000 dynes That is 1 lb = 445 000 dynes

Note 2 In deciding upon the invariant unit mass in the fps system we calculate the mass in grams which would be given an acceleration of 30 48 cm/sec/sec by a force of 445,000 dynes as if the units were to apply to an inertial or absolute system

$$\text{unit mass} = \frac{445\,000}{30\,48} = 14{,}600 \text{ g} \quad \text{or 1 slug}$$

* Assuming that the body is in a locality where g = 981 cm/sec/sec

† Where g = 9 81 m sec/sec

‡ Where g = 32 19 ft/sec/sec.

been classed, however arbitrarily, as fundamental units. In this connection it should be noted that, in the American foot-pound-second (fps) system of units, the pound (a unit of force) is considered a fundamental unit as shown in Table II.

The three units selected as fundamental are three of the primary units of the system, and all other primary units are derived from these three fundamental units, employing defining equations similar to those which have been discussed in this chapter. The primary units of several physical quantities in each of three systems of units are listed in Table II. It will be observed that only a few of the derived units have been given special names. Generally speaking, the primary units of the derived quantities in a given system of units are reckoned in terms of the units of the three quantities which have been selected as fundamental quantities.

In the study of electricity and magnetism, the student will progress more surely and more rapidly if he learns how to form the primary units of the derived quantities directly from the fundamental units than if he attempts to memorize a large number of specially named units.

For any set of fundamental units, there exists one complete set of primary units, and with respect to this system all other units are either secondary or hybrid units. Little or no attention need be given the matter of units in theoretical analyses which are wholly general in character because any set of primary units is applicable. Many analyses, however, cannot be carried to a successful conclusion without introducing one or more numerical coefficients into the expressions at some stage of the development; and it is precisely at this point that the expression (or expressions) lose generality and become restricted to the one set of units to which the numerical coefficients apply.

The International Electrotechnical Commission in 1935 unanimously approved the adoption of the meter, kilogram, and second as the fundamental units of length, mass, and time. From what has been said about the arbitrary selection of fundamental units, it should be plain that, as applied to mechanics at least, the mks system possesses no particular points of superiority over any other metric system that employs units of reasonable size. The mks system of units, however, possesses certain points of superiority over other systems of units in the field of electricity and magnetism. This system will be used almost exclusively in this text and will be discussed in greater detail at appropriate places throughout the text.

7. Secondary Units. The engineering profession employs a large number of secondary units in order to avoid specifying a particular magnitude as a very small fraction of a primary unit or as a very large number of these primary units. Secondary units are usually indicated by prefixes

which specify the magnitudes as decimal fractions of or decimal multipliers of the primary unit. Some of these prefixes are shown below where the term (meter) is employed simply for purposes of illustration and it is understood that (meter) might be replaced with any other primary unit.

"One-millionth part of" ( ) is abbreviated *micro*(meter).
"One-thousandth part of" ( ) is abbreviated *milli*(meter).
"One-hundredth part of" ( ) is abbreviated *centi*(meter).
"One thousand" ( ) is abbreviated *kilo*(meter).
"One million" ( ) is abbreviated *meg*(meter).

This list is far from complete, and various combinations of the prefixes are employed. For example "one-millionth-millionth (10^{-12}) part of" is specified as *micromicro* or as $\mu\mu$, and "one-thousandth-millionth (10^{-9}) part of" is sometimes abbreviated *millimicro*. The latter abbreviation appears on one type of impedance bridge which is widely used in practice.

Although secondary units are employed in practice to specify the magnitudes of particular quantities, these secondary units are seldom used directly in derived equations to obtain a result. In general, conversion to primary units must be made before the quantity is entered numerically into a derived equation if the result is desired in primary units.

8. Transformation or Conversion of Units. Although the mks system of units will be used almost exclusively in this text, it should be plain that the adoption of one set of units can never obviate the necessity for transformation of units until all measuring devices are changed to read directly in the primary units of the adopted system. Since this ideal state of affairs will never exist, some transformations or conversions of units are inevitable, and a few of the problems in this text are designed specifically to test the reader's ability to make transformations of the kind normally encountered in practice.

The reader is undoubtedly familiar with the fact that the relative sizes of units are stated in a variety of different ways in various handbooks. The following expressions might, for example, be found, and they are all simply different ways of stating the same physical fact, namely, that the *meter* (m) is a larger unit of length than the *foot* by the factor 3.281.

$$\frac{\text{ft}}{\text{m}} = 3.281 \quad \text{or} \quad \frac{\text{m}}{\text{ft}} = 0.3048 \quad \text{(as algebraic ratios)}$$

and

$$1 \text{ m} \equiv 3.281 \text{ ft} \quad \text{or} \quad 1 \text{ ft} \equiv 0.3048 \text{ m} \quad \text{(as identities)}$$

where the symbol $\equiv$ means "identically equal to."

The symbol $\equiv$ is employed to distinguish the statement "1 ft is identically equal to 0.3048 m" from the somewhat different statement that implies "the *number* of meters per foot is equal to 0.3048." In order to appreciate the distinction between the two statements one should test out certain types of transformations of units which are relatively unfamiliar so that common sense will not automatically dictate the correct procedure.

As an example of the type of transformation which is often encountered in the study of electricity and magnetism, consider the working equation for magnetomotive force ($\mathcal{F}$) which, in the unrationalized ab-cgs system of units,[2] reads

$$\mathcal{F} = 4\pi N I_{\text{ab}} \quad \text{gilberts} \tag{4}$$

where N is the number of turns in the current-carrying coil (which sets up $\mathcal{F}$) and I_{ab} is the magnitude of this current expressed in abamperes, a unit of current which is larger than the practical (or mks) unit of current, the ampere, by the factor 10. That is,

$$\frac{\text{abamp}}{\text{amp}} = \frac{1}{10} \quad \text{(as a ratio)}$$

or

$$1 \text{ abamp} \equiv 10 \text{ amp} \quad \text{(as an identity)}$$

Let the problem be that of writing equation (4) in such a manner that I_{ab} is replaced by I_{amp} while holding the units of $\mathcal{F}$ constant. If it is recognized that I_{ab} in equation (4) implies "*number* of abamperes carried by the turns of wire," then it is convenient to write out the required transformation as

$$\mathcal{F} = 4\pi N\left(\frac{I_{\text{amp}}}{10}\right) = 0.4\pi N I_{\text{amp}} \quad \text{gilberts} \tag{5}$$

since the *number* of abamperes (I_{ab}) $= \frac{1}{10} \times$ *number* of amperes (I_{amp}).

In other words, the required transformation is obtained simply by direct algebraic substitution for I_{ab} by its equivalent $I_{\text{amp}}/10$ as obtained from the specified *ratio*. The fact that the required transformation is here accomplished by the *ratio method* does not imply that it cannot be done equally well starting with the statement

$$1 \text{ abamp} \equiv 10 \text{ amp}$$

But in order to arrive at the correct result when this identity is used, a person is likely to do somewhat more detailed thinking than if he employs

[2] The difference between rationalized and unrationalized magnetic units will be considered later.

the ratio method because I_{ab} in equation (4) refers to the "*number* of abamperes" and an inversion of the factor 10 as it appears in the above identity must be made before a direct substitution can be made. Nevertheless, many engineers prefer the identity method of stating the relative sizes of units because it gives them a clear conception of the relative sizes.

In the final analysis, the best check that can be made on a transformation of units is to visualize clearly the relative sizes of the units involved and then proceed to exercise caution and common sense in making the transformation. If, however, the units involved are not well-known units or if they are compound units, some reliable algebraic method should be adopted.

Example. For the sake of further illustration, consider the defining equation for mechanical energy (or work) in a simple case where **f** and **l** (as vectors) are directed along the same line. Then $\cos \theta]_{\mathbf{l}}^{\mathbf{f}} = \cos 0° = 1$, and

$$W = fl$$

where f is constant at a specified value of 48 ounces (48 oz)
l is a fixed distance or displacement of 2 meters (2 m).
The problem is simply that of specifying W in pound-feet and in newton-meters, the primary units of energy in the fps and mks systems of units respectively. Plainly

$$W = 48 \times 2 = 96 \text{ oz-m} \quad \text{(of energy or work)}$$

but the *ounce-meter* is a *hybrid unit* of energy which is not acceptable here.
The necessary data which are readily obtainable are

$$\frac{\text{oz}}{\text{lb}} = 16 \qquad \frac{\text{ft}}{\text{m}} = 3.281 \qquad \frac{\text{newtons}}{\text{lb}} = 4.45 \quad \text{(as } \textit{ratios}\text{)}$$

or

$$1 \text{ oz} \equiv \tfrac{1}{16} \text{ lb} \quad 1 \text{ ft} \equiv 0.3048 \text{ m} \quad 1 \text{ newton} \equiv 0.225 \text{ lb} \quad \text{(as } \textit{identities}\text{)}$$

The *newton* is the primary unit of force in the mks system, and as shown above it is smaller than the pound by the factor 0.225. More precisely 1 newton $\equiv 10^5$ dynes, but ordinarily three-significant-figure accuracy is employed here.

1. *The Ratio Method.* As applied to the transformation of f,

$$f_{oz} = 48 \text{ oz}$$

transforms into

$$f_{lb} = \tfrac{48}{16} = 3 \text{ lb}$$

by multiplying f_{oz} by the ratio lb/oz = 1/16. Thus

$$f_{lb} = (f_{\cancel{oz}}) \times \left(\frac{\text{lb}}{\cancel{\text{oz}}}\right) = (48) \times \left(\frac{1}{16}\right) = 3 \text{ lb}$$

and the cancellation marks indicate clearly that the result is in pounds because the (48) is dimensionally ounces and the $\frac{1}{16}$ is dimensionally lb/oz. Similarly,

$$l_{\text{m}} = 2\text{ m}$$

transforms into

$$l_{\text{ft}} = 2 \times 3.28 = 6.56\text{ ft}$$

by multiplying l_{m} by the ratio ft/m. That is,

$$l_{\text{ft}} = (l_{\cancel{\text{m}}}) \times \left(\frac{\text{ft}}{\cancel{\text{m}}}\right) = (2) \times (3.28) = 6.56\text{ ft}$$

The desired result is obtained as

$$W = f_{\text{lb}} \times l_{\text{ft}} = 3.0 \times 6.56 = 19.68 \quad \text{lb-ft} \quad \text{(of energy)}$$

or

$$W = f_{\text{newton}} \times l_{\text{m}} = \left(f_{\cancel{\text{oz}}} \times \frac{\cancel{\text{lb}}}{\cancel{\text{oz}}} \times \frac{\text{newtons}}{\cancel{\text{lb}}}\right) \times l_{\text{m}}$$

$$= (48 \times \tfrac{1}{16} \times 4.45) \times 2 = 26.7 \quad \text{newton-m}$$

The newton-meter is the primary unit of energy in the mks system and is often called a *joule.*

As a further illustration of the ratio method of transformation and as a check on the above arithmetic, one might observe that

$$\frac{W_{\text{lb-ft}}}{W_{\text{newton-m}}} = \left(\frac{\text{lb}}{\text{newton}}\right) \times \left(\frac{\text{ft}}{\text{m}}\right) = (0.225) \times (3.28) = 0.738$$

and hence

$$W_{\text{lb-ft}} = (W_{\cancel{\text{newton-m}}}) \times \left(\frac{W_{\text{lb-ft}}}{W_{\cancel{\text{newton-m}}}}\right) = (26.7) \times (0.738) = 19.68 \quad \text{lb-ft}$$

2. *The Identity Method.* This method arrives at the same result as the ratio method but does so by disregarding the dimensional features of the ratio. Instead, each of the units to be transformed is changed directly into the desired unit by considering it to be a certain multiple of the desired unit. For example, each of the 48 oz of force of the present problem is transformed into $\frac{1}{16}$ lb by the identity 1 oz $\equiv \frac{1}{16}$ lb as shown below.

$$W_{\text{lb-ft}} = (48 \times \tfrac{1}{16}\text{ lb}) \times (2.0 \times 3.28_{\text{ft}}) = 19.68 \quad \text{lb-ft}$$

or

$$W_{\text{newton-m}} = (48 \times 0.278_{\text{newton}}) \times 2_{\text{m}} = 26.7 \quad \text{newton-m}$$

3. *Comparison of the Two Methods.* Unless the reader has had occasion to make some of the more subtle types of unit transformations encountered in the engineering profession, he is likely to think that the present subject is "much ado about nothing." But after he has encountered such physical quantities as *thermal conductivity* expressed in "g-cal per sec per sq cm per deg cent per cm" he will appreciate the importance of the subject. Experience

has shown that students invariably have trouble with unit transformations, and undoubtedly much of the confusion stems from the fact that the two methods of transformation are identical numerically but somewhat different in concept.

It is impossible to say that one method is better than the other because with a slightly different type of maneuvering either method arrives at the same place. The ratio method seems to be somewhat better adapted to a first course because, with the aid of the cancellation marks, the writer can show the reader precisely what has been done where a succession of transformations like

$$(f_{\cancel{ft}})\left(\frac{\text{newtons}}{\cancel{lb}}\right)\left(\frac{\cancel{lb}}{\cancel{ft}}\right) = f_{\text{newtons}}$$

are written.

The details of all future transformations of a simple nature will be neglected, and the reader may make these transformations by either method if he cares to check the results which are given. In a few places where the reader may experience difficulty with the transformations, the details will be given consistently in terms of the *ratio method* in order to avoid confusion between the two methods.

MATHEMATICAL MANIPULATIONS

9. Vector Addition. A vector quantity (like length, area, velocity, or force) is not completely specified unless its magnitude, which is a scalar

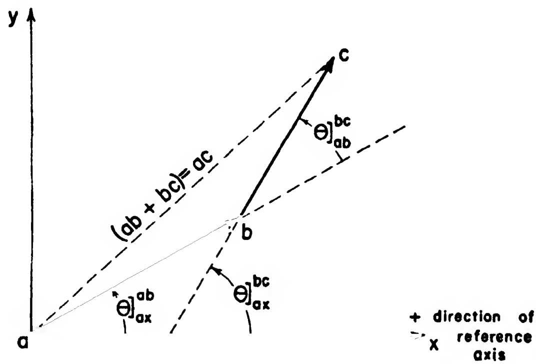

FIG. 2. Illustrating the addition of length **ab** and **bc**.

quantity, and its direction with respect to some reference direction (or axis) are stated.[3] Length **ab** in Fig. 2, for example, is not completely specified unless both the magnitude and direction of the length are stated. One method of writing coplanar vectors with respect to a given

[3] Bold-face type is used here to designate vector quantities where the directional properties of these quantities are of importance.

reference axis (ax) is suggested in Fig. 2. That is

$$\mathbf{ab} \text{ (as a vector)} = ab\underline{/\theta\rbrack_{\mathbf{ax}}^{\mathbf{ab}}} \tag{6}$$

where the right side of the equation indicates that the magnitude of the vector **ab** is ab unit lengths, and the associated angle, $\underline{/\theta\rbrack_{\mathbf{ax}}^{\mathbf{ab}}}$, indicates that the vector **ab** is displaced (in a counterclockwise direction) from the reference axis (**ax**) by θ degrees.

Like vectors which are added or subtracted (in accordance with the rules of vector algebra) yield vector quantities of the same species as the original vectors. If, for example, length **bc** in Fig. 2 is added to **ab**, the result is length **ac**, and the manipulations involved in effecting this vector addition are shown below.

Example. In Fig. 2, let

$$\mathbf{ab} = 10\underline{/30^\circ} \quad \text{and} \quad \mathbf{bc} = 8\underline{/60^\circ} \quad \text{unit lengths}$$

The x axis component of the vector **ac** is plainly

$$\Sigma x = 10 \cos 30^\circ + 8 \cos 60^\circ = 12.66 \quad \text{unit lengths along } x \text{ axis}$$

The y axis component of the vector **ac** is

$$\Sigma y = 10 \sin 30^\circ + 8 \sin 60^\circ = 11.93 \quad \text{unit lengths along } y \text{ axis}$$

The magnitude of the vector **ac** is then

$$ac = \sqrt{12.66^2 + 11.93^2} = 17.4 \quad \text{unit lengths}$$

and the direction of the vector **ac** with respect to the **ax** direction is

$$\theta\rbrack_{\mathbf{ax}}^{\mathbf{ac}} = \tan^{-1} \frac{11.93}{12.66} = 43.3^\circ$$

Hence

$$\mathbf{ac} = 17.4\underline{/43.3^\circ} \quad \text{unit lengths}$$

In general, two or more vectors may be added together in terms of the summation of the x axis and y axis components. If the three forces $\mathbf{f}_1$, $\mathbf{f}_2$, and $\mathbf{f}_3$ of Fig. 3 are considered as acting at point O, then

$$\Sigma x = Of_1 \cos \alpha_1 + Of_2 \cos \alpha_2 + Of_3 \cos \alpha_3$$

$$\Sigma y = Of_1 \sin \alpha_1 + Of_2 \sin \alpha_2 + Of_3 \sin \alpha_3$$

The resultant force acting at point O is the sum of Of_1, Of_2, and Of_3, or

$$\mathbf{Of}_R = \sqrt{(\Sigma x)^2 + (\Sigma y)^2}\ \underline{/\tan^{-1} \frac{\Sigma y}{\Sigma x}} \tag{7}$$

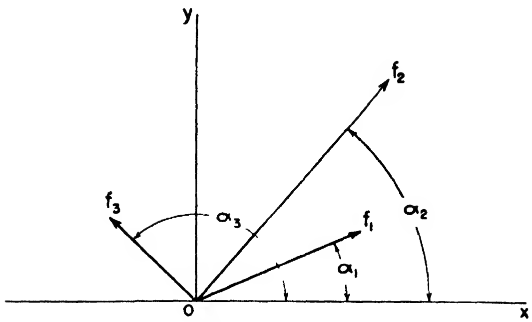

FIG. 3. Illustrating three forces $\mathbf{f}_1$, $\mathbf{f}_2$, and $\mathbf{f}_3$ acting at point O.

10. Dot (or Scalar) Product of Two Vectors. Vector quantities may be multiplied in two distinctly different ways. The reader may accept the rules for the multiplication of vectors which are given here on the basis that, if the rules are followed, the result obtained will be a physical quantity which is useful in explaining electric and magnetic phenomena. It will become more apparent later that some of the basic laws and concepts of electricity and magnetism can be written in general form much more concisely if vector notation is employed than if ordinary scalar notation is employed.

The product of two vectors may be taken in such a manner that the result is a scalar quantity. Probably the most common example of this type of multiplication is the product of a vector force and the vector length (or displacement) through which the force acts. If this product is to represent work, we realize that only the component of force **f** which is directed along the **l** path is to be included in the calculations. In order to signify this fact in abbreviated style, we write

$$\text{work} = \mathbf{f} \cdot \mathbf{l} = (f \cos \theta]_{\mathbf{l}}^{\mathbf{f}})\,(l) = fl \cos \theta]_{\mathbf{l}}^{\mathbf{f}} \tag{8}$$

where $\mathbf{f} \cdot \mathbf{l}$ is called the *dot* or *scalar* product of **f** and **l**.

The rules for evaluating the dot product are clearly indicated by the right-hand member of equation (8) where the symbols f and l refer only to the magnitudes of the vectors **f** and **l**.

Example. In Fig. 4, an outside agent exerts a force of 500 dynes on a physical body which is constrained in some manner to move along the **ol** direction rather than directly along the direction of the applied force. The work done by the outside agent in moving the body 10 cm along the **ol** direction is

$$\text{work} = (500 \cos 30°)(10) = 4330 \quad \text{dyne-cm} \quad \text{or} \quad \text{ergs}$$

Or, reduced to mks units,

$$\text{work} = \left(500\cancel{\text{dynes}} \times \frac{\text{newtons}}{\cancel{\text{dyne}}} \times \cos 30^\circ\right)\left(10\cancel{\text{cm}} \times \frac{\text{m}}{\cancel{\text{cm}}}\right)$$

$$= (500 \times 10^{-5} \times \cos 30^\circ)(10 \times 10^{-2})$$

$$= 4330 \times 10^{-7} \quad \text{newton-m} \quad \text{or} \quad \text{joules}$$

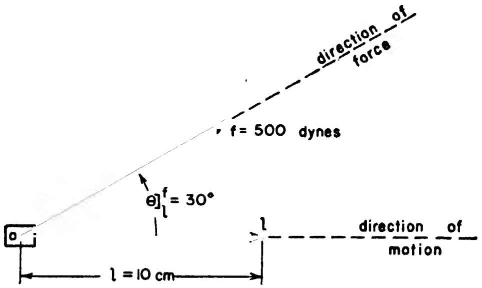

FIG. 4. Vector **f** directed at 30° from vector **l**.

11. Cross (or Vector) Product of Two Vectors. The product of two vectors may be taken in such a manner that a *vector* quantity results from the indicated multiplication. This type of vector product is seldom used as such in elementary courses but it is widely used in advanced courses. The only aspect of the *vector* product which is of immediate concern to us here is that of representing an area (**A**) as a physical vector quantity in such a way that the direction of the *vector area* is clear to the reader. (It will be remembered that, for any vector quantity to be completely identified, both the *magnitude* and *direction* of the vector must be specified.)

The *cross* or *vector* product provides us with a convenient and clear-cut method of specifying both the magnitude and direction of an *area.* As applied to the two vector lengths **ab** and **bc** of Fig. 5, the *cross* product of the two vector lengths is *by definition*

$$\mathbf{ab} \times \mathbf{bc} = \mathbf{A} = \left\{(ab)(bc)\sin\theta\Big]_{\mathbf{ab}}^{\mathbf{bc}}\right\}\underline{/90^\circ} \qquad (9)$$

where the magnitude of vector $\mathbf{A} = (ab)(bc)\sin\theta\Big]_{\mathbf{ab}}^{\mathbf{bc}} = A$

the direction of vector **A** is *normal to the face* of the scalar area A, or normal to the plane in which vectors **ab** and **bc** lie.

The rules for actually evaluating the magnitude of the *cross* product in terms of the scalar magnitudes of the two vectors are shown clearly by the right-hand member of equation (9), but so far as we are concerned here this magnitude might have been evaluated by any other

method which would yield the correct value of the scalar area. The advantage to be gained by the cross product notation is that we have a rule for specifying the *direction* of this area.

The sense or direction of the vector area is *by definition* at right angles to the plane in which vectors **ab** and **bc** are located and is customarily taken as the direction in which a right-hand screw would travel if a line (or slot) on the head of the screw coinciding initially with vector **ab** [the first term in equation (9)] were turned to coincide with the direction of the **bc** vector. This rule as applied to Fig. 5 implies that the right-hand screw would be turned through the smaller angle, $\theta]_{\mathbf{ab}}^{\mathbf{bc}}$, and hence the screw would travel *out* of the page. The *direction* of the vector area **A** is *by definition* normal to the xy plane and pointing toward the reader in this case.

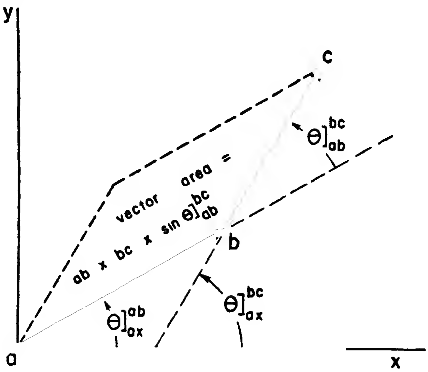

FIG. 5. Illustrating vector area as determined by the cross product **ab × bc**.

Without the aid of cross-product notation, the visualization or the specification of an area *as a vector quantity* is rather awkward. The advantage of using this notation will become more apparent when the concept of magnetic or electric flux threading through a vector area at various angles is considered.

Example. A concept which will be repeatedly encountered in the study of magnetism is

$$\text{magnetic flux } \phi = \mathbf{B} \cdot \mathbf{A} = BA \cos \theta]_{\mathbf{B}}^{\mathbf{A}} \quad \text{units of flux}$$

where **B** (a vector quantity) is equal to the magnetic flux density
A (a vector quantity) is equal to, say, **ab × ac** in Fig. 6.

If the above equation is in agreement with the physical facts (and it is), the fact that ϕ is equal to the *dot* product of **B** and **A** makes magnetic flux a scalar quantity in accordance with the rules of vector algebra. But in the evaluation of this *dot* product it becomes necessary to know the angle between the vector **B** and the vector **A**. This angle is indicated in Fig. 6 as the angle θ.

If the proper interpretation of the *cross* product is made of **ab × ac** in Fig. 6, we note that the magnitude of **A** is simply $(ab)(ac)$ since the angle between the two sides (**ab** and **ac**) of the rectangle is 90°. [See equation (9).] The direction of the vector area **A** (for use in $\phi = \mathbf{B} \cdot \mathbf{A}$) is *normal to the face* of the area, that is, normal to the plane in which the sides **ab** and **ac** are located.

In a case where the magnetic flux density vector (**B**) is also normal to the face of the area, the θ angle shown in Fig. 6 is zero, and

$$\phi = BA \cos \theta = BA \quad \text{(maximum flux threading through area } \mathbf{A}\text{)}$$

If, however, the **B** vector lies in the same plane as do the sides **ab** and **ac**, the θ angle in Fig 6 will be 90°, and

$$\phi = BA \cos 90° = 0 \quad \text{(zero flux threading through area } \mathbf{A}\text{)}$$

In a more general case, the θ angle might have any specified value, and the result would be evaluated accordingly. A little thought will show that without the aid of cross-product notation for the vector **A**, a great many words and diagrams might be required to bring out the full generality of the statement $\phi = \mathbf{B} \cdot \mathbf{A}$.

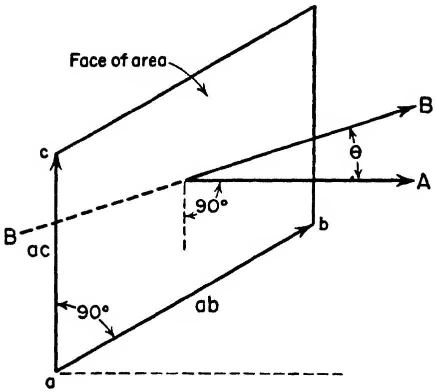

FIG. 6. Illustrating a vector area **A** pierced by a vector **B**.

12. Work (or Energy) Expressed as a Line Integral. A concept which is of fundamental importance in the study of electricity and magnetism is that of the *line integral* in a vector field. If it were possible to do so, we would defer until later the concept of the line integral. Some of the basic notions of electric potential and energy levels, however, require at least that the physical meaning of the line integral be clearly understood. Formal evaluations of these integrals are of relatively little importance since approximate results can always be obtained by simple arithmetical methods if the physical meaning of these integrals is appreciated.

The line integral which is of immediate importance to us is that which represents the work done in moving a body (or possibly an electrical charge) from one point to another point in a field of force. The understanding here is that the body (or possibly the electrical charge) experiences a force of some specified description at each point in the field or the region containing the two points. We begin by joining the two points (say A and B) by some arbitrary curve. This curve or path will be regarded as being composed of individual line elements or segments each of length $d\mathbf{l}$.

The infinitesimal amount of work done on a body in moving it an infinitesimal distance $d\mathbf{l}$ along the specified path is

$$dW = \mathbf{f} \cdot d\mathbf{l} = (f \cos \theta]_{d\mathbf{l}}^{\mathbf{f}})(dl) \qquad (10)$$

where ($f \cos \theta]^{f}_{dl}$) is the component of the applied force which is in the direction of the vector dl. If f is a function of l (that is, varies along the path of motion) or if $\cos \theta$ is not constant, the total work done in moving a body from A to B can be evaluated precisely as

$$W = \int_A^B f \cos \theta]^{f}_{dl}\, dl \tag{11}$$

The above integral is one form of line integral, and it is the formal way of summing up all the individual dW's that are encountered as the body moves from point A to point B along the path or *line* of motion. In this connection A and B are physical distances (or lengths) measured from a common reference point. (See Fig. 7.)

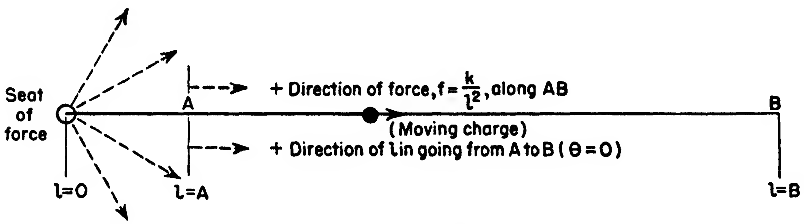

FIG. 7. Illustrating direction of force and direction of motion.

In Fig. 7, the seat of force might be an electrical charge which exerts a force f on a like charge which moves from $l = A$ to $l = B$ under the influence of force f. Throughout the journey from A to B, the moving charge has a force exerted on it which is a function of l, that is, $f = k/l^2$, where k is some constant. In other words, the force varies inversely as the square of the distance from the seat of force.

Even though the detailed mechanism of obtaining a result from the operations indicated on the right-hand side of equation (11) is not fully appreciated at this stage, the reader can undoubtedly appreciate the physical significance of

(1) calculating $f \cos \theta]^{f}_{\Delta l}$ at the center of each of several equal and finite sections of the path of motion, assuming that the total length of the path is broken up into several Δl's.

(2) multiplying $f_1 \cos \theta]^{f}_{\Delta l_1}$ by Δl_1 to obtain the approximate value of

$$W_1 = (f_1 \cos \theta]^{f}_{\Delta l_1})(\Delta l_1);$$

(3) repeating (2) for each of the Δl's into which the path of motion has been divided;

(4) summing up the results of (2) and (3) to obtain an approximate value of the total work $W = \sum \Delta W$.

The larger the number of Δl's chosen, the smaller will be the value of each Δl and the more nearly accurate will be the result. In using equation (11), we divide the path up into an infinite number of infinitesimally short segments each dl in length, and it is by this process that an accurate result is obtained. The physical meaning of a line integral in a vector field is of fundamental importance to the whole of mathematical physics including the study of electricity and magnetism.

Example. *Evaluation of Line Integral.* If, in Fig. 7, $l = A = 0.75$ m and $l = B = 4.75$ m, the work done on the moving charge by the seat of force might be evaluated with the aid of equation (11) as

$$W = \int_A^B \left(\frac{k}{l^2}\cos 0\right) dl = k\left[-\frac{1}{l}\right]_{l=0.75}^{l=4.75} = k\left[\frac{1}{0.75} - \frac{1}{4.75}\right] = 1.122k$$

For the conditions which have been specified, $1.122k$ is the correct result.

Approximate Solution. A fairly good approximation to the correct result may be obtained if we divide the path of motion up into 8 equal segments, evaluate the incremental amount of work done in moving over each of these segments, and sum the 8 incremental values of work done to obtain the total work W. For $l = A = 0.75$ m and for $l = B = 4.75$ m, the total length of path is 4.0 m, and each of the segments is 0.5 m in length.

The segment (or Δl) nearest the seat of force has its center at $l = 1.0$ m, and the force which acts at this point is $k/l^2 = 1.0k$. The incremental amount of work done in moving over the first 0.5-m segment is then approximately equal to $(1.0k)(0.5) = 0.50k$ units of work. The details of the eight sets of calculations are shown in tabular form below.

Interval or Segment from $l = A$	Value of l at Segment Center	Value of $f = k/l^2$	Value of ΔW $(0.5f)$
1	1.00	$1.000k$	$0.500k$
2	1.50	$0.444k$	$0.222k$
3	2.00	$0.250k$	$0.125k$
4	2.50	$0.160k$	$0.080k$
5	3.00	$0.111k$	$0.055k$
6	3.50	$0.082k$	$0.041k$
7	4.00	$0.062k$	$0.031k$
8	4.50	$0.049k$	$0.025k$
			$W = 1.079k$

In this particular case, the approximate value obtained for $W(1.079k)$ is about 4 per cent in error. Greater accuracy may be obtained by choosing smaller incremental lengths than 0.5 m. (See Prob. 15 at the close of the chapter.)

PROBLEMS

1. (*a*) What is the primary unit of power in the mks system of units expressed in terms of the fundamental units of this system?

(*b*) What other names are sometimes used to designate this unit of power?

2. A physical body, the mass of which is 200 g, possesses an instantaneous velocity of 60 mph. What are the instantaneous values of (*a*) velocity, (*b*) momentum, and (*c*) kinetic energy of this body, all expressed in mks units?

3. Express a length of 0.015 cm in millimicrometers.

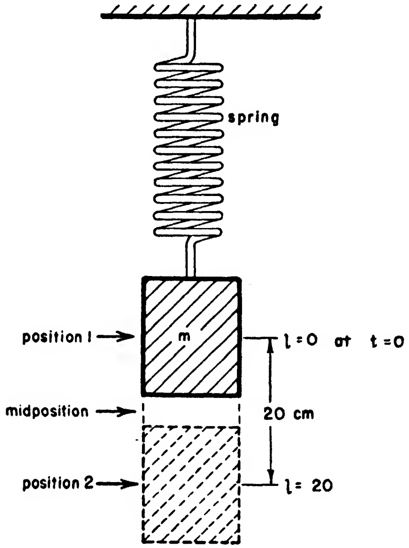

FIG. 8. See Prob. 4.

4. Consider the case of a mass m oscillating between limiting positions as shown in Fig. 8. If l is reckoned as zero when the mass is in position 1, the algebraic expression for l as a function of time is

$$l = 10\left[1 - \cos\left(\frac{2\pi}{4}t\right)\right] \quad \text{cm}$$

where $2\pi/4$ is the angular velocity of oscillation in radians per second provided t is expressed in seconds.

(*a*) What is the value of l 1 sec after $t = 0$; 2 sec after $t = 0$?

(*b*) What is the instantaneous velocity at $t = 0$, $t = 0.5$ sec, and $t = 1$ sec?

(*c*) What is the time-averaged value of velocity over the 1-sec interval which starts at $t = 0$ and ends at $t = 1$ sec?

5. The algebraic expression for the velocity of a moving body (expressed as a function of time t) is

$$v = (9t^2 + 20) \quad \text{m/sec} \qquad 0 < t < 3 \text{ sec}$$

(*a*) What is the instantaneous velocity at $t = 2$ sec?

(*b*) What is the time averaged value of velocity over the 1-sec interval which starts at $t = 1$ sec and ends at $t = 2$ sec?

(*c*) What is the distance traversed by the body from $t = 0$ to $t = 2$ sec?

6. Find the *number* of gram-calories (of heat energy) per British thermal unit, employing the ratio method of conversion:

$$\frac{\text{g-cal}}{\text{Btu}} = \frac{\text{g} \times {}^\circ\text{C}}{\text{lb} \times {}^\circ\text{F}} = \frac{\text{g}}{\text{lb}} \times \frac{{}^\circ\text{C}}{{}^\circ\text{F}}$$

It is known that 1 lb $\equiv$ 453.6 g, or that g/lb = 453.6, and that 1°C $\equiv$ 1.8°F. Which is the larger unit of heat energy, the gram-calorie or the Btu?

7. It is known from experimental evidence that in transformations between mechanical energy and heat energy, 1 Btu $\equiv$ 778 lb-ft. Find the *number* of newton-meters per Btu or the number of joules per Btu using the ratios

$$\frac{\text{newtons}}{\text{lb}} = 4.45 \quad \text{and} \quad \frac{\text{m}}{\text{ft}} = 0.3048$$

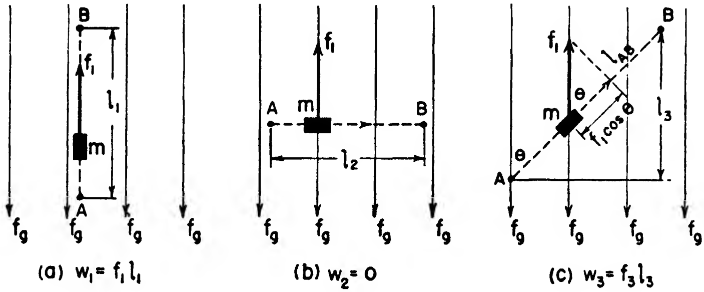

FIG. 9. See Prob. 13.

By the ratio method, one may readily visualize the transformation in the form

$$\frac{\text{newton-m}}{\text{Btu}} = \frac{\cancel{\text{lb}}\text{-}\cancel{\text{ft}}}{\text{Btu}} \times \frac{\text{newtons}}{\cancel{\text{lb}}} \times \frac{\text{m}}{\cancel{\text{ft}}}$$

Which is the larger unit of energy, the newton-meter or the Btu?

8. Two forces ($\mathbf{F}_1$ = 30 newtons and $\mathbf{F}_2$ = 40 newtons) acting at a point are displaced from one another by 120°. What is the magnitude and direction of the resultant force $\mathbf{F}_R = \mathbf{F}_1 + \mathbf{F}_2$? Specify the direction of $\mathbf{F}_R$ relative to the direction of $\mathbf{F}_1$.

9. Given two vectors ($\mathbf{B} = 100\ \underline{/30^\circ}$ webers/sq m and $\mathbf{A} = 0.20\ \underline{/0^\circ}$ sq m) where the associated angles are specified with respect to an arbitrary reference axis. Find the dot product $\mathbf{B} \cdot \mathbf{A}$ in accordance with the rules of vector algebra and specify the units of the product thus formed. *Note*: Even though the name of the unit of the product be meaningless to the reader at this stage, he should be able to name the unit from the specified data.

10. Construct a parallelogram which has two connecting sides equal respectively to

$$\mathbf{oa} = 5\ \underline{/0^\circ}\text{ cm} \quad \text{and} \quad \mathbf{ob} = 2\ \underline{/60^\circ}\text{ cm}$$

Find the vector areas which are defined by

$$\mathbf{oa} \times \mathbf{ob} = \mathbf{A}_1 \quad \text{and} \quad \mathbf{ob} \times \mathbf{oa} = \mathbf{A}_2$$

In what respect is $\mathbf{A}_1$ different from $\mathbf{A}_2$?

11. How much work does a man accomplish (in a scientific sense) by holding a 5-lb weight at arm's length for a period of 2 min?

12. Show that (mass × acceleration) is dimensionally equivalent to W^1l^{-1}.

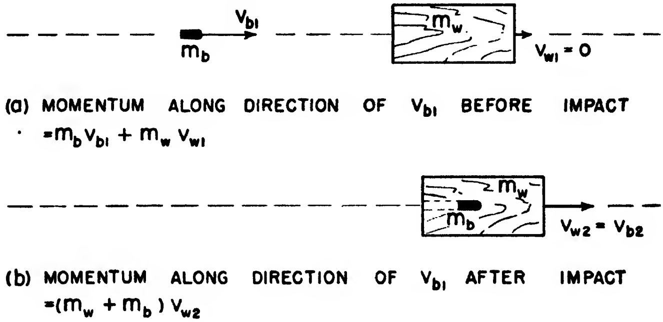

FIG. 10. Illustrating the conservation of linear momentum. See Prob. 16.

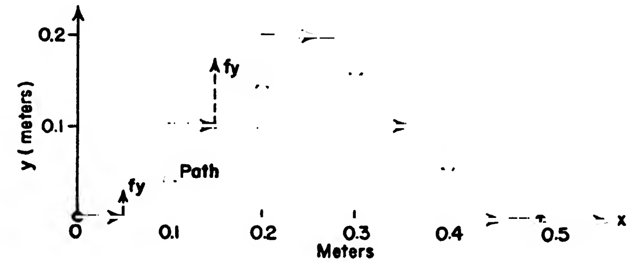

FIG. 11. See Prob. 17.

13. In Fig. 9 are shown three different displacements of a mass ($m = 2$ kg) in the earth's gravitational field. Remembering that

$$\text{force of gravity on a mass } m = mg$$

where g is 9.81 m/sec/sec in mks units, find the work done in

(a) moving m from A to B through the distance $l_1 = 5$ m
(b) moving m from A to B through the distance $l_2 = 5$ m
(c) moving m from A to B through the distance $l_{AB} = 7.07$ m if $\theta = 45°$.

14. A body travels 1 m in going from point A to point B. Along the direction of travel, the force varies uniformly from $f_A = 0$ at point A to $f_B = 0.10$ newton at point B. What amount of work is done upon the body in going from A to B if it is assumed that the direction of the force and the direction of motion coincide at all points along the path of motion?

15. Refer to the approximate evaluation of the line integral in the example on page 20. Make the same type of evaluation employing 20 segments (rather than 8) and compare the accuracy of this 20-segment evaluation with that obtained by the 8-segment evaluation shown at the close of Section 12.

16. The wooden block m_w in Fig. 10-a has a mass of 1 kg and an initial velocity which is equal to zero. The mass of the bullet, m_b, is 0.02 kg, and the velocity of $(m_w + m_b)$ just after impact in Fig. 10-b is observed to be 9.8 m/sec.
What is the velocity of the bullet just before impact?

17. In Fig. 11 is shown a field of force which, for x expressed in meters, is defined by

$$\mathbf{f}_y = 100x \text{ newtons} \quad \text{(directed along the } y \text{ axis direction)}$$

It is assumed that $\mathbf{f}_y$ is the force acting upon a hypothetical particle which is moved by an outside agent from $x = 0$ to $x = 0.5$ m along the path indicated. It is further assumed that $\mathbf{f}_y$ is the only force acting upon the particle along the path of motion.
Find the *net positive work done by the outside agent* in moving the particle from $x = 0$ to $x = 0.5$ m.

CHAPTER II

Versatility of Electrical Charge

In order to acquaint the reader with the versatility of electricity, several aspects of electricity or of electrical charge are reviewed rather briefly in this chapter, and experimental evidence is the only justification given for the various electrical effects which are described. Beginning with Chapter III, a more theoretical and systematic study of electricity is presented.

Electricity (or electrical charge) is, in a broad sense, a physical agent used by engineers to transform and convert energy from one form to another. Electrical charge, of and by itself, is not energy in the accepted sense. As will be shown later, the separation of electrical charge into its positive and negative constituents results in potential energy of a form that can be readily converted into other forms of energy.

1. Electrical Charge as a Source of Potential Energy. Electrical charge is a fundamental property of matter and is here classified as a fundamental physical quantity. Electrically neutral bodies are composed of equal amounts of positive and negative charge; and bodies which are electrically neutral obey the ordinary laws of mechanics. It is well known, however, that "electrified" bodies do not obey these laws.

Two hard rubber rods, for example, which have been rubbed briskly with flannel or fur will repel each other in a noticeable manner because each of the rods has acquired an excess of negative charge. Two glass rods which have been rubbed with silk will also repel each other, but in this case each of the rods has acquired an excess of positive charge as a result of the rubbing process. One of the hard rubber rods and one of the glass rods (which have been electrified by rubbing) will exhibit a force of attraction for one another which is millions of times greater than the gravitational force of attraction which exists between the two rods when they are electrically neutral. In order to account for these phenomena, matter was endowed with the property of being charged positively and negatively even before individual positive and negative charges were identified experimentally.

Coulomb, about 1785, measured the forces of attraction and repulsion between charged bodies accurately enough to generalize the results of his measurements. (See Fig. 1.) For two bodies which were small in diameter (as compared with their distance of separation) he found that

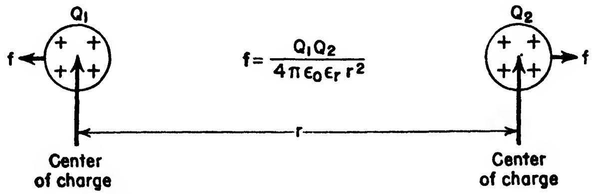

FIG. 1. Centers of charge, centers of mass practically coincident.

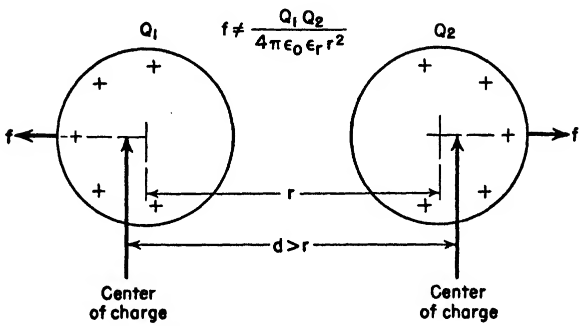

FIG. 2. Centers of charge, centers of mass not coincident.

(1) The force varied directly as the product of the two charges.

(2) The force varied inversely as the square of the distance of separation, provided this distance was great compared with the physical dimensions of the bodies.[1]

(3) The force depended upon the medium or dielectric in which the bodies were immersed, being less for material media than for free space.

The above results led directly to the formulation of Coulomb's inverse

[1] The fact that like charges repel each other (and unlike charges attract) is responsible for this qualification. If the small metallic spheres shown in Fig. 1 are sufficiently widely separated, the mutual force of repulsion does not appreciably disturb the positive charge that is uniformly distributed over the surfaces of the spheres. The "centers of charge" are, therefore, practically coincident with the centers of mass so r is well defined.

Under the conditions shown in Fig. 2, however, the distance r is relatively small and does not represent accurately the distance between the centers of charge. The mutual forces of repulsion of the positive charges cause these charges to drift apart somewhat as shown in Fig. 2. In a case of this kind, the distance between the centers of charge can be determined only by advanced methods.

square law which, in equation form and in *rationalized* units, reads as follows:

$$f = \frac{Q_1 Q_2}{4\pi\epsilon_0 r^2} \quad \text{(for free space or air)} \tag{1}$$

where f represents the mutual force of repulsion or attraction
Q_1 and Q_2 are the magnitudes of the two charges in question
r is the distance of separation between the two "centers of charge"
ϵ_0 is the permittivity of free space being equal to $1/(36\pi \times 10^9)$ in *rationalized* mks units and equal to $1/4\pi$ in *rationalized* cgs units.

The permittivity of free space is that property of free space which permits the mutual forces of attraction or repulsion to exist and as such is a physical quantity being dimensionally equivalent to $Q^2 l^{-2} f^{-1}$.

The distinction between *rationalized* units and *unrationalized* units is that ϵ_0 in rationalized units is only $1/4\pi$ times as large numerically as is the ϵ_0' of unrationalized units. In other words, $4\pi\epsilon_0 = \epsilon_0'$. Since we propose to use rationalized units throughout this text, we are forced to use $4\pi\epsilon_0$ in equation (1) rather than the unrationalized value ϵ_0' which is often used in connection with Coulomb's inverse square law. The use of $4\pi\epsilon_0$ in this little-used equation will eliminate the presence of 4π in many of the widely used working equations which will be encountered later. Various other reasons might be cited for employing rationalized units. (See Preface.)

If other than free space (or air) is employed as a dielectric medium, Coulomb's inverse square law takes the form

$$f = \frac{Q_1 Q_2}{4\pi\epsilon_0\epsilon_r r^2} = \frac{Q_1 Q_2}{4\pi\epsilon r^2} \tag{2}$$

where ϵ_r is the *relative* permittivity (or dielectric constant) of the dielectric medium in which the charges are immersed.

The numerical values of ϵ_r for several of the more common dielectrics are given in Table I. ϵ_r expresses only the ratio of f in free space to the corresponding value of f where a material dielectric is employed, and as such is a numeric or dimensionless quantity. The product $\epsilon_0\epsilon_r$ is commonly written as simply ϵ in general equations as indicated in equation (2).

The fact that unlike charges attract each other with appreciable force and are ever ready to recombine (if given an opportunity) is of basic importance to the electrical engineer. If by some means he can separate + charge from − charge he has a source of potential energy which may be so controlled that (during the recombination of the + and − charges)

the potential energy initially present is converted into useful forms of energy. With suitable circuits and other devices, this potential energy may, during the recombination period, be transformed into mechanical energy, heat energy, radiant energy like light, or even into chemical energy.

2. Units of Charge. The three most commonly used units of electrical charge are the *coulomb*, the *statcoulomb*, and the electronic charge or, simply, the *electron*.

The *coulomb* is defined as that unit of charge which, if placed 1 m from a similar charge in free space, is repelled with a force of 9×10^9 newtons. It is the primary unit of charge in the mks system of units and is of convenient size for all ordinary circuit calculations. That the coulomb is a ridiculously large unit of charge for use in Coulomb's inverse square law is of minor importance. This will become more apparent after we have considered the fact that *1 coulomb* of charge falling through *1 volt* of electric potential difference develops *1 joule* (or newton-m) of energy.

TABLE I

Substance	Relative Permittivity (Dielectric Constant)
Air (atm. pr., 0°C)	1.00058
Paraffin wax	1.9 to 2.3
Dry paper	2.0 to 3.5
Rosin	2.5
Shellac	2.7 to 3.7
Wood (many varieties)	3.0
Impregnated paper	3 to 4
Mica	4 to 8
Glass	4 to 10
Porcelain	4 to 9
Quartz	4.6 to 4.7
Water (distilled)	80 to 81.5

The student of electrical engineering either has encountered or will encounter in other textbooks the statcoulomb as a unit of charge. The *statcoulomb is defined* as that unit of charge which, if placed 1 cm from a similar charge in free space, is repelled with a force of 1 dyne (the primary unit of force in cgs units). This unit of charge is widely used by physicists and is often employed by engineers in theoretical derivations which in one way or another involve the application of Coulomb's inverse square law.

The *electron* is the ultimate or indivisible unit of negative electrical charge because pertinent experimental evidence supports the view that all negative charges in the universe consist simply of integral or whole

numbers of electronic charges. The magnitude of the electronic charge has been found experimentally to be -1.6×10^{-19} coulomb or -4.8×10^{-10} statcoulomb. Hence

$$\frac{\text{(No. of) electrons}}{\text{coulomb}} = \frac{1}{1.6 \times 10^{-19}} = 6.25 \times 10^{18} \quad \text{(as a ratio)}$$

or

$$\frac{\text{(No. of) electrons}}{\text{statcoulomb}} = \frac{1}{4.8 \times 10^{-10}} = 2.08 \times 10^{9} \quad \text{(as a ratio)}$$

Example. Let it be required to find the relative size of the statcoulomb and the coulomb from the force definitions which have been given. Reference to equation (1) will show that in any rationalized system of units

$$Q = \sqrt{4\pi\epsilon_0 r^2 f}$$

In stat-cgs units:

$$Q_{\text{statcoulombs}} = \sqrt{4\pi \frac{1}{4\pi} \times \text{cm}^2 \times \text{dynes}}$$

In mks units:

$$Q_{\text{coulombs}} = \sqrt{4\pi \frac{1}{36\pi \times 10^9} \times \text{m}^2 \times \text{newtons}}$$

As a ratio:

$$\frac{\text{statcoulombs}}{\text{coulombs}} = \sqrt{9 \times 10^9 \times \frac{\text{cm}^2}{\text{m}^2} \times \frac{\text{dynes}}{\text{newton}}}$$

$$= \sqrt{9 \times 10^9 \times 10^4 \times 10^5} = 3 \times 10^9$$

The above ratio carries with it the understanding that we are evaluating the *number of* statcoulombs per coulomb, and the result obtained indicates that the coulomb is a much larger unit of charge than the statcoulomb since there are 3×10^9 statcoulombs per coulomb.

Had the experimental specifications of the *number of* electrons per coulomb and per statcoulomb been used, we would have arrived at the same result, namely,

$$\frac{\text{statcoulombs}}{\text{coulomb}} = \frac{\cancel{\text{electrons}}}{\text{coulomb}} \times \frac{\text{statcoulombs}}{\cancel{\text{electron}}}$$

$$= 6.25 \times 10^{18} \times \frac{1}{2.08 \times 10^9} = 3 \times 10^9$$

3. Law of Superposition as Applied to Coulomb Forces. Assuming that there are more than two charges in a locality, the resultant force on any one charge may be calculated as the *vector* sum of the forces caused by the other charged bodies taken one at a time. This is simply an application of a general principle called the law of superposition

which states that if, under all conditions, *effects* (forces in this case) *remain directly proportional to causes* (charges in this case), then the resultant effect may be determined as the resultant of the effects calculated one at a time.

Example. Three concentrated charges Q_1, Q_2, and Q_3 are located at the vertices of an equilateral triangle as shown in Fig. 3. In this connection it is convenient to think of the charges as residing on small mass particles which, except for the assigned positive charges, are electrically neutral. Let it be required to find the resultant force on the charge Q_3.

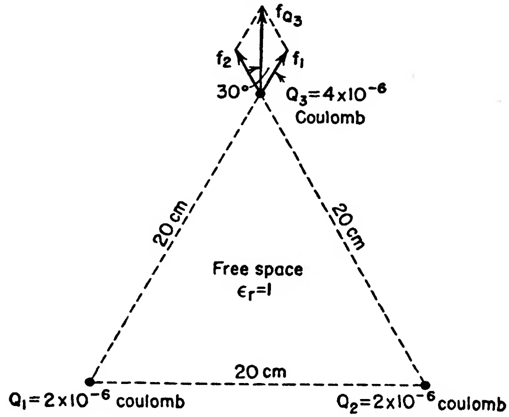

FIG. 3. Illustrating the superposition principle.

The component of the resultant force which is caused by the mutual repulsion of Q_1 and Q_3 is, by equation (1),

$$f_1 = \frac{(2\times 10^{-6})(4\times 10^{-6})}{4\pi \times \dfrac{1}{36\,\pi\times 10^9}\times 0.2^2} = \frac{(9\times 10^9)(8\times 10^{-12})}{0.04} = 1.8 \quad \text{newtons}$$

Since, in this case, the magnitude of $\mathbf{f}_2$ is numerically the same as the magnitude of $\mathbf{f}_1$, the resultant force on Q_3 may be found as

$$\mathbf{f}_{Q_3} = \mathbf{f}_1 + \mathbf{f}_2 = (2\,f_1 \cos 30°)\underline{/90°} = 3.12\underline{/90°} \quad \text{newtons}$$

where the $\underline{/90°}$ specifies the direction of $\mathbf{f}_{Q_3}$ relative to the direction of the line which joins Q_1 and Q_2.

4. The Role of Electricity in a Country's Greatness. Power is the time rate of doing work or of expending energy and has been defined in Chapter I as $p = W/t$. Industrially speaking, a country is *powerful* if it possesses the ability to do a large amount of work in a short period of time. The United States is a powerful nation because it produces more

than 300 billion kilowatt-hours[2] of energy annually. Practically all the energy used industrially comes from the work done by falling water or from the combustion of fuels. About 70 per cent of this energy is obtained from fuel-driven generators and 30 per cent from hydro plants.

In electrical form, the energy may be transmitted over great distances and then reconverted into the desired form of energy at the receiving station. The desired form might, for example, be light, heat, mechanical energy, or chemical energy.

The most important type of electrical generator from an industrial point of view is the one in which a prime mover (like a steam turbine or a water wheel) drives copper conductors through a strong magnetic field. When a conductor sweeps across a magnetic field, some of the electrons within the conductor are forced to one end, thus charging this end of the wire (or conductor) negatively. When electrons are forced to one end of the conductor (negative terminal), the other end of the conductor becomes positively charged because some of the electrons have been forced to leave what becomes the positive terminal of the generator. The desired energy transformation or conversion then takes place when the electrons recombine (through suitable circuits) with the positive charge that is present at the positive terminal.[3] This brief description of the electrical generator will, of course, have more meaning to the reader after he has considered the electrical structure of matter and has studied the laws of magnetism.

An illustration of the manner in which large amounts of work can be accomplished with the aid of electrical motors is given in Fig. 4. The detailed work cycle of this washing and crushing establishment is unimportant here; the fact that work of this kind can be controlled by push buttons at a central station is of interest. About 85 per cent of our industrial machinery is driven electrically.

Example. A strong man might, conceivably, lift with a force of 100 lb through a distance of 4 ft once each minute throughout a 10-hr day; in which case he would perform 240,000 lb-ft of work in the course of the 10 hr day.

Let it be required to express this amount of energy in kilowatt-hours and to determine its cost in terms of electrical energy which can be purchased for 2 cents per kilowatt-hour.

[2] The abbreviation for kilowatt-hour is kwhr; it is the unit in which electrical energy is normally sold by public utility companies, the price ranging from about 0.7 to 7 cents per kilowatt-hour; 300 billion kilowatt-hours is the equivalent of 10.8×10^{17} joules or 8.0×10^{17} lb-ft of energy.

[3] The reader is reminded here of the principle of conservation of energy, namely, that energy can neither be created nor destroyed but only altered in form. This law was enunciated in the nineteenth century and has been the plague of inventors of perpetual-motion machines ever since.

The 240,000 lb-ft of energy will first be changed into newton-meters of energy because this transformation is presumably straightforward in light of the transformations of units which have been made in Chapter I. (See page 12.)

$$(\cancel{\text{lb-ft}})\left(\frac{\text{newton-m}}{\cancel{\text{lb-ft}}}\right) = (240{,}000)(1.356) = 326{,}000 \quad \text{newton-m}$$

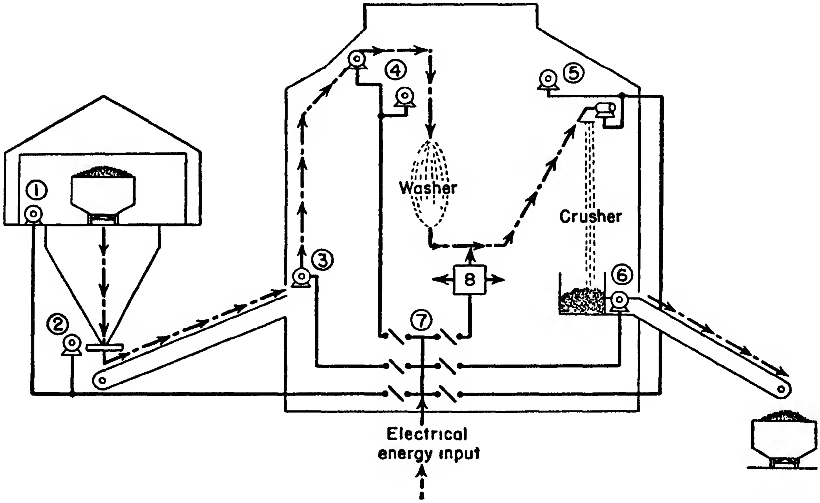

FIG. 4. Transformation of electrical energy into mechanical energy. (1) Motor driving rotary car dump. (2) Motor driving feeder. (3) Motor driving elevating conveyer. (4) Conveyer and washer motors. (5) Conveyer and crusher motors. (6) Conveyer motor. (7) Central push button control station. (8) Lighting panel board.

The newton-meter, the primary unit of energy in the mks system, is also known as the joule, or as the watt-second. The particular name applied to this unit of energy depends somewhat on the type or form of energy involved; for example, the newton-meter applies more specifically to mechanical energy or work whereas the watt-second is most frequently used in specifying amounts of electrical energy.

$$326{,}000 \text{ newton-m} \equiv 326{,}000 \text{ watt-sec}$$

and

$$(\cancel{\text{watt-sec}})\left(\frac{\text{kwhr}}{\cancel{\text{watt-sec}}}\right) = (326{,}000)\left(\frac{1}{3{,}600{,}000}\right) = 0.0905 \quad \text{kwhr}$$

At 2 cents per kwhr:

$$\text{cost of energy} = 0.0905 \times 2 = 0.181 \quad \text{cent}$$

If the man's work could have been done with electrical motors, a considerable monetary saving could obviously have been effected.

5. The Role of Electricity in Communications. Telegraphy, telephony, radio, and television are all forms of communication in which the intelligence being communicated is transmitted electrically from a sending station to a receiving station.

In voice transmission, for example, the mechanical energy possessed by the voice or sound waves is first converted into electrical form at the sending end by means of a telephone transmitter or radio microphone. These devices serve the same general purpose as the electrical generator described in the preceding section; but in the present case, the power involved is in the order of milliwatts, whereas the generator's power capacity would likely be in the hundreds or thousands of kilowatts.

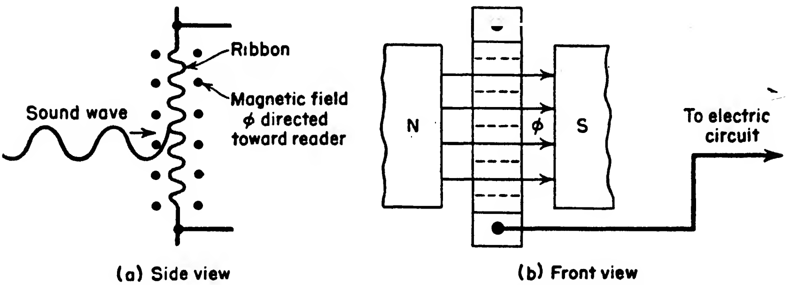

FIG. 5. Illustrating the operating principle of the velocity microphone.

The velocity (or ribbon) microphone in fact operates on the same general principle as the large electrical generator in that the movement of a conductor (say a duralumin ribbon) is made to cut across the magnetic field of a permanent magnet.[4] (See Fig. 5.) In this manner the sound wave is transformed into a corresponding electrical variation and transmitted as such over wires (in the case of the ordinary telephone) or through the atmosphere (in the case of radio). At the receiving station, the transmitted electrical variations are transformed back into sound waves when the electrical variations are passed through the telephone receiver or the radio loudspeaker.

A relatively new form of communication which gained prominence during World War II and which is now finding several commercial uses is *radar*. In radar, a *succession of signals* of very great power and of very short duration is sent out as a pulsed radio wave, reflected from the

[4] Various other types of transmitters are in common use as, for example, the ordinary carbon granule telephone transmitter where the voice wave causes the carbon granules to exhibit lower and higher electrical resistance in accordance with the condensations and rarefactions of this wave. Other types of microphones commonly used are the condenser and crystal microphones but the detailed operation of these devices will not be considered at this point.

target back to the sending station, and there received as a *succession of signals* which are much reduced in power but which can be interpreted to give the shape and location of the target relative to the sending position. Its principal commercial uses at the present time are in the navigation of ships and in ground landing approaches for airplanes.

That the field of communications is "big business" may be judged from the following figures which were roughly correct as of 1948 with every indication then that appreciable increases were inevitable.

5,000	regular broadcast stations in the world	35% in U.S.A.
58,000,000	telephones in the world	60% in U.S.A.
135,000,000	radio receivers in the world	50% in U.S.A.

THE ROLE OF ELECTRICITY IN CHEMISTRY AND PHYSICS

6. Elementary Particles. A reasonably clear mental picture of the structure of matter is essential in many phases of electrical engineering because elementary particles (usually sub-atomic in form) are the charge carriers which are employed to accomplish the desired energy transformations previously referred to in this chapter. In the few pages that are here devoted to the electrical structure of matter, only a few of the basic notions can be given, and these mostly in qualitative form. It is not essential at this point that the reader appreciate the full significance of the quantitative examples which are given, but a perusal of these examples may help to strengthen his mental picture of the electrical structure of matter.

There is a vast accumulation of experimental evidence which indicates that all matter is (or can be) built up from a few elementary building blocks. The three most important building blocks are the *electron,* the *proton*, and the *neutron*.

7. The Electron. The electron as previously stated is considered to be the ultimate or indivisible unit of negative electrical charge. For most engineering purposes, the electron may be considered as an electrically charged mass particle, the mass and charge of which are

$$m_e = 9.1 \times 10^{-31} \text{ kg} \quad \text{or} \quad 9.1 \times 10^{-28} \text{ g}$$

$$Q_e = -1.6 \times 10^{-19} \text{ coulomb} \quad \text{or} \quad -4.8 \times 10^{-10} \text{ statcoulomb}$$

In striving for a theory which is consistent with observed radiation effects (spectrum lines and the like), physicists have endowed the electron with wavelike properties which can be understood only through the medium of quantum or wave mechanics, a subject which is well beyond the scope of this text. Fortunately, the wavelike properties of

the electron are of little immediate importance in the field of electrical engineering.

8. The Atom. All atoms are about 10^{-10} m in diameter and consist of a central positively charged nucleus, about 10^{-14} m in diameter, surrounded by enough electrons to make the atom electrically neutral. The chief constituents of an atom are shown in Table II. As a result

TABLE II

Name of Atomic Constituent	Electrical Charge in Coulombs (to 2 significant figures)	Mass (kg)	Mass Units (as used by nuclear physicists) (equivalent to 1.6603×10^{-27} kg)
Electron	-1.6×10^{-19}	9.1×10^{-31}	about 0.000547
Proton	$+1.6 \times 10^{-19}$	1.6729×10^{-27}	1.00758
Neutron	0	1.6751×10^{-27}	1.00893

of experimental information obtained during nuclear physics experiments throughout World War II, the masses of the proton and the neutron are now known accurately to about five or six significant figures.[5]

The Proton. The proton is a constituent particle of an atomic nucleus, having a *positive* charge which is numerically equal to an electronic charge and a mass which is about 1840 times that of an electron. The proton itself *is* the nucleus of an ordinary hydrogen atom (H) as pictured in Fig. 6. In general, however, the nucleus of an atom contains one or more neutrons in addition to one or more protons.

The Neutron. The neutron is a basic constituent particle of atomic nuclei having no electrical charge. It possesses a mass which is slightly greater than the mass of the proton. (See Table II.) Whether the neutron is a close union of an electron and proton or a totally distinct particle is not precisely known at the present time.

Mass Number. The *mass number* of an atom (written H^1 for a hydrogen atom, He^4 for a helium atom, Li^7 for a lithium atom, on up to U^{238} for a heavy uranium atom) *refers to the number of massive particles, protons and neutrons, in the nucleus of the atom.* Atomic nuclei which are heavier than U^{238} have been produced artificially. Examples of these heavier nuclei are neptunium, Np^{239}, and plutonium, Pu^{239}. These two new chemical elements differ from one another in that neptunium has 93 orbital electrons whereas plutonium has 94.

[5] The mass units of the proton and neutron as they appear in Table II are taken from "A General Account of the Development of Methods of Using Atomic Energy for Military Purposes under the Auspices of the United States Government," written by H. D. Smyth in 1945 and obtainable from the Superintendent of Documents, Washington 25, D.C.

Atomic Number. The atomic number is an integer characteristic of each chemical element which tells how many protons there are in the atomic nucleus and also how many electrons there are in the atom, outside the nucleus. For example, the atomic number of the helium atom is 2 since the nucleus contains two protons and there are two orbital electrons outside the nucleus. The atomic numbers of several of the

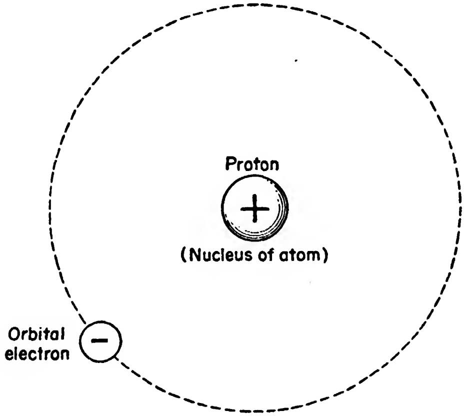

FIG. 6. Schematic representation of a hydrogen atom.

more common chemical elements are shown in Appendices B and C. *Atomic number is not to be confused with mass number.*

9. Atomic Structure. It is now generally accepted that the atoms of the 92 chemical elements found in nature (hydrogen, helium, lithium, · · ·, uranium) differ from one another by reason of the number and arrangement of protons, neutrons, and orbital electrons that make up the atom. The common conception is that one or more electrons move about a positively charged nucleus, and that under normal conditions the nucleus constitutes practically the entire mass of the atom. The nucleus possesses a positive charge which is equal in magnitude and opposite in sign to the combined charge of the orbital electrons. Thus the normal atom is so constituted that it appears to be electrically neutral to outside bodies even though forces of electrical attraction and repulsion are operative within the confines of the atom itself.

The orbital electrons may be thought of as revolving about the nucleus in an orbit which is approximately 10^{-10} m in diameter. Since the nucleus is about 10^{-14} m in diameter (or smaller) and since the electrons are also very small in comparison with the orbit, whose diameter is 10^{-10} m, the atom is relatively speaking a very open type of structure. In this respect the atom is something like the solar system.

Schematic representations of four common atomic structures are shown in Fig. 7. An orbital electron possesses a definite potential energy by virtue of its position with respect to the positively charged nucleus. The electron possesses kinetic energy of rotation by reason of its motion

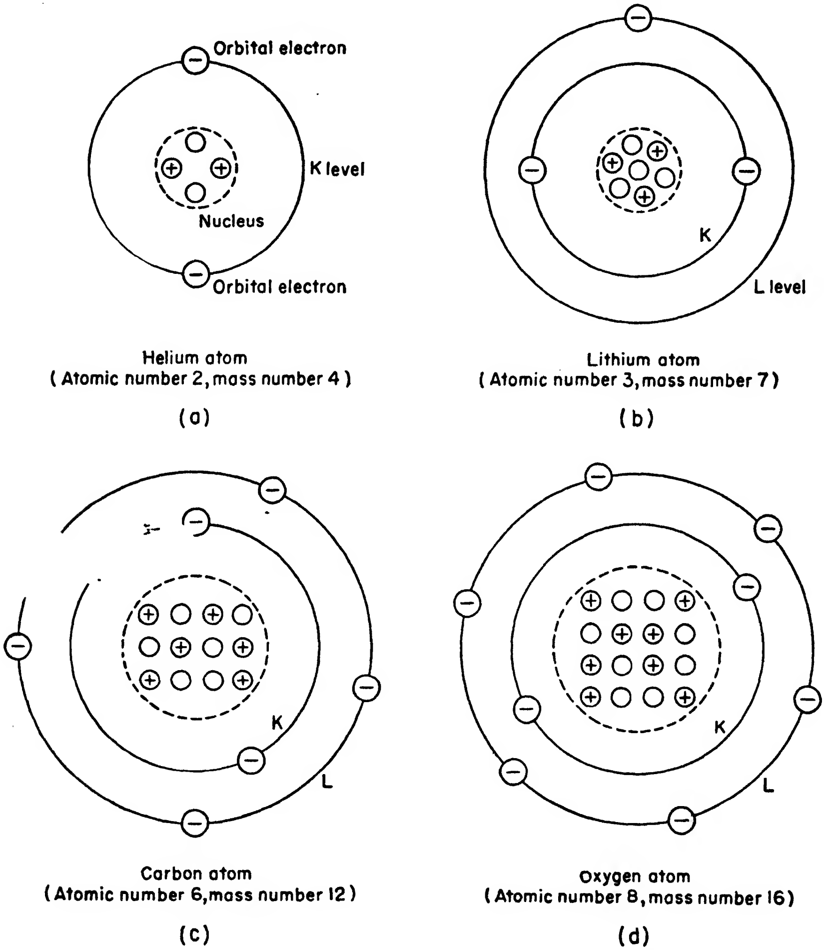

FIG. 7. Schematic representation of some common atoms.

about the nucleus. It is not difficult to visualize a normal atom as one in which the electric force of attraction (between nucleus and electron) is balanced by the centrifugal (mv^2/r) force to form a state of mechanical equilibrium. (See Fig. 6.) This picture is, however, oversimplified because it fails to indicate why the orbital electrons are confined to specified orbits and only those orbits. (The discreteness of the orbits is

usually explained in more advanced courses than this one in terms of Planck's "quantum of action.")

Example. *Energy Levels of Orbital Electrons.* As applied to a single proton ($+Q_e$) and a single electron ($-Q_e$), the potential energy of the electron may be defined as *the work done by an outside agent* in moving the electron from an infinitely great distance (where its potential energy is zero) to a distance r from the proton.

The force that will have to be exerted (by the outside agent) at any distance x from the proton is $f' = Q_e^2/4\pi\epsilon_0 x^2$. This force is equal in magnitude but opposite in direction to the inverse-square-law force of attraction between $+Q_e$ and $-Q_e$. As indicated in Fig. 8, the direction of motion is in the minus-x

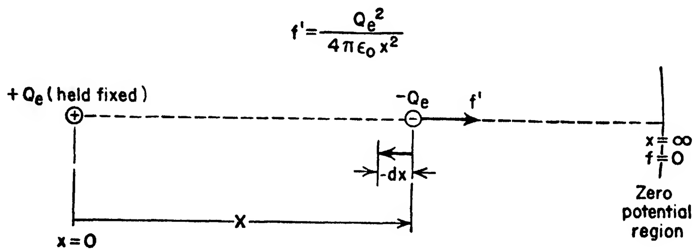

FIG. 8. Determination of potential energy of $-Q_e$.

direction since the origin of x is taken at the proton position. The angle between f' and the incremental displacement ($-dx$) is 180°. The incremental amount of work done by the outside agent in moving $-Q_e$ through an incremental distance is

$$dW_{me} = f'(-dx) \cos \theta\Big]_{f'}^{dx} = f' \, dx$$

in accordance with the definition of work.

The potential energy of the electron at $x = r$ can be found by summing up all of the dW_{me}'s between $x = \infty$ and $x = r$. The most satisfactory way of performing this summation is by means of integral calculus. (See page 19.) Let W_p represent the potential energy. Then

$$W_p = \int_{x=\infty}^{x=r} f' \, dx = \frac{Q_e^2}{4\pi\epsilon_0} \int_{\infty}^{r} \frac{dx}{x^2} = \frac{Q_e^2}{4\pi\epsilon_0}\left[-\frac{1}{x}\right]_{\infty}^{r} = -\frac{Q_e^2}{4\pi\epsilon_0 r} \tag{3}$$

If the electron fell into the field of the proton without the outside agent's guidance and revolved in a circular path, its kinetic energy of rotation would be

$$W_k = 0.5 m_e v^2 \tag{4}$$

where v is the linear velocity in the circular path which has been assumed.

Balance between the centrifugal (mv^2/r) and the inverse-square-law attraction demands that

$$\frac{m_e v^2}{r} = \frac{Q_e^2}{4\pi\epsilon_0 r^2} \tag{5}$$

If the above equation is multiplied through by $r/2$ and the result compared with equation (4), it is found that

$$W_k = \frac{m_e v^2}{2} = \frac{Q_e^2}{8\pi\epsilon_0 r} \tag{6}$$

The energy possessed by the electron traveling in an orbit of r radius is the sum of its potential and kinetic energy, or

$$W = W_p + W_k = -\frac{Q_e^2}{4\pi\epsilon_0 r} + \frac{Q_e^2}{8\pi\epsilon_0 r} = -\frac{Q_e^2}{8\pi\epsilon_0 r} \tag{7}$$

The negative sign implies that positive energy will have to be imparted to the electron to remove it from the orbit of r radius.

The elementary analysis given above shows what is meant by an orbital electron possessing energy and why a definite amount of energy ($Q_e^2/8\pi\epsilon_0 r$ units) would have to be supplied (say by an outside agent in the form of a high-speed particle) to strip the hydrogen atom of its orbital electron. The analysis, however, does not show why only discrete values of r in equation (7) actually occur in nature.

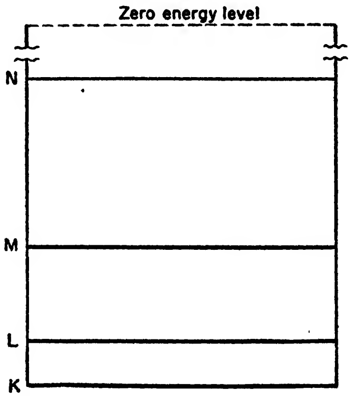

FIG. 9. Representation of K, L, M, and N energy levels.

For our purposes, it will be sufficient to accept the fact that only particular values of r and hence of W, in equation (7), can actually exist in nature and that these values of W are called energy levels. The primary energy levels that exist in atomic structure are called the K, L, M, N, O, P, and Q levels. The K, L, M, and N levels are diagrammed qualitatively in Fig. 9. In accordance with our zero-energy reference, the K level is the "greatest negative" level and is usually referred to as the "lowest" level. The L level is the next higher level, and so on. An electron originally in the K level must receive some positive or additional energy from an outside source to be moved to the L level, and still more positive energy to be moved to the M level.

The number of electrons in the various energy levels of the first 38 chemical elements is shown in Appendix C. The chemical properties of all elements depend upon the number of orbital electrons and the arrangement of these electrons in the various energy levels. Valence, stability, and tendency to combine with other atoms are the properties which can be predicted from the orbital arrangements shown in Appendix C.

These topics, however, will not be considered at this time. The brief discussion of atoms which has been given in this section will have served its purpose if it has provided the reader with a qualitative picture or representation of the electrical structure of matter.

10. Ionized Atoms. An atom may lose one or more of its electrons in any one of several different ways.

(1) A fast-moving electron or other particle may collide with a normal atom and completely dislodge one or more electrons from the outer orbit of the atom. In this case the energy required to dislodge the orbital electrons comes from the kinetic energy of the moving particle.

(2) The neutral atoms of an acid, base, or salt molecule may dissociate in the presence of water in such a way as to become either positively or negatively charged. If upon dissociation an atom loses an electron it becomes positively charged, and if it gains an electron during the dissociation process it becomes negatively charged.

An atom which has either lost or gained one or more electrons is said to be in an *ionized* state. If an atom loses or gains one electron it possesses a net charge of 1.6×10^{-19} coulomb and is called a *singly charged* ion. If two electrons are lost or gained the net charge is 3.2×10^{-19} coulomb, and the atom is referred to as a *doubly charged* ion. Under the influence of electric forces, ionized atoms can be made to move through liquid and gaseous conductors, thus transporting electrical charge through these media.

11. Excited Atoms as a Source of Radiant Energy. An atom which has all its orbital electrons (as specified in Appendix C) but with one or more of these electrons in a "higher-than-normal" energy level is said to be in an *excited* state. An orbital electron in a "higher-than-normal" energy level usually returns very quickly (in much less than a microsecond) to its normal state. In returning to the lower energy level, the electron releases energy, and this released energy takes the form of radiant energy which may be within the visible spectrum.[6] The range of the visible spectrum is shown in Fig. 10.

An example of an excited atom is illustrated in Fig. 11 where e' is the orbital electron which is momentarily raised to a "higher-than-normal" level by the kinetic energy of electron e. In returning to its normal level, electron e' releases energy which is in the ultra-violet portion of the spectrum. This ultra-violet radiant energy may be employed in fluorescent lamps to energize the fluorescent coating of these lamps which

[6] Radiant energy having frequencies ranging from 4×10^{14} to 7.5×10^{14} cycles/sec (or wave lengths ranging from 0.75×10^{-6} to 0.4×10^{-6} m in free space) is capable of affecting our eyes. Radiant energy outside this narrow band of frequencies (or wave lengths) cannot be "seen."

in turn emits radiant energy that is distributed throughout the visible spectrum.

The propagation of radiant energy may be visualized as a wave motion in space; much the same as the propagation of a disturbance on the surface of still water. The chief characteristic of radiant energy is

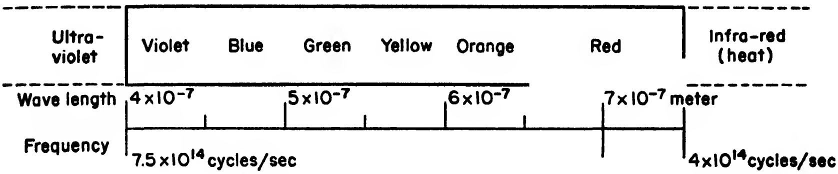

FIG. 10. The visible spectrum.

that it is propagated rectilinearly through space at a constant velocity. In free space (or air) this velocity is very nearly 3×10^8 m/sec.

The peculiarity of an excited atom is that it can emit radiant energy of only discrete frequencies since the electrons can occupy only discrete energy levels within the atomic structure. When an electron of an

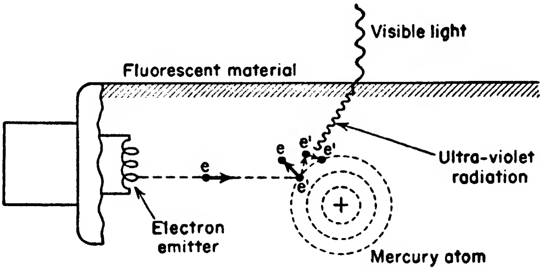

FIG. 11. An excited mercury atom emits ultra-violet radiation.

excited atom returns to its normal energy level, it releases a quantized amount of radiant or electromagnetic energy. This energy is released in only definite quantities because the electrons can occupy only definite energy levels within the atomic structure. The fundamental relationship which governs this type of radiation is

$$W_q = W_i - W_f = hf \tag{8}$$

where W_q is the change in energy
W_i is the initial energy and
W_f the final energy
h is Planck's constant, 6.542×10^{-27} erg-sec or 6.542×10^{-34} joule-sec
f is the frequency of the electromagnetic energy radiated.

Since the wave length of all radiant energy is related to the frequency by the velocity of light, equation (8) indirectly defines the wave length of the radiated energy in terms of the change of energy which produces the radiation. That is,

$$\lambda f = c \quad \text{or} \quad \lambda = \frac{c}{f} \tag{9}$$

where λ is the wave length, f the frequency, and c the velocity of light. This relationship applies to all electromagnetic wave propagation including radio waves, heat, light, X-rays, and cosmic rays, provided the propagation occurs in free space (or air) and is not confined to restricted channels like wave guides.

Example. Reference to Appendix C will show that a sodium atom has 1 electron in the M energy level. In an excited condition this electron is permitted by nature (and by wave-mechanic equations) to occupy momentarily an energy level which is 2.09 electron-volts (or 3.344×10^{-19} joule) higher than normal. This excited electron returns to its normal M energy level in perhaps a hundredth of a microsecond and in so doing releases 2.09 electron-volts or 3.344×10^{-19} joule of energy.

The electron-volt is a unit of energy which is widely used in specifying energy levels of electrons and in specifying other small quantities of energy. It is smaller than the joule (the primary unit of energy in the mks system) by the same factor that an electronic charge is smaller than the coulomb. As a ratio,

$$\frac{\text{electron-volts}}{\text{joule}} = \frac{1}{1.6 \times 10^{-19}} = 6.25 \times 10^{18}$$

The significance of this ratio will become more apparent later. For the present, an electron-volt may be considered simply as a unit of energy which is 6.25×10^{18} times smaller than the joule.

From the fundamental relationship stated in equation (8), it follows that

$$f = \frac{3.344 \times 10^{-19}}{6.542 \times 10^{-34}} = 5.11 \times 10^{14} \quad \text{cycles/sec}$$

From equation (9):

$$\lambda = \frac{3 \times 10^{8}}{5.11 \times 10^{14}} = 5.87 \times 10^{-7}\ \text{m}$$

The reciprocal of the wave length, λ, gives the number of waves per unit length. In this case the number of waves per meter of the radiated energy is 1,700,000. The wave length represented by 5.87×10^{-7} m is within the visible spectrum and is the characteristic sodium yellow. (See Fig. 10.)

The example given above, although based on experimental data, is rather oversimplified in the way in which the data are presented. The actual behavior of an excited atom is a complex and fascinating subject

which has been considered briefly here because it shows how even the color of light can be traced to an electrical origin.

12. Combination of Atoms as a Source of Heat Energy. A large percentage of our industrial energy comes from the combustion of coal, a process in which the carbon atoms of the coal combine with diatomic molecules of oxygen to release heat energy. The process may be written in the form of a chemical equation as

$$C + O_2 \rightarrow CO_2 + \text{heat energy} \tag{10}$$

where the CO_2 appears in the flue gas.

The reaction stated in equation (10) occurs only if the temperature of the molecules is sufficiently great, but when the reaction does occur some energy in the form of heat is released. This released energy is explained on the basis that the sum of the internal binding energies of the carbon atom (C) and of the diatomic molecule of oxygen (O_2) taken separately is greater than the internal binding energy of the CO_2 molecule. Hence, by the law of conservation of energy, some energy is released, and this released energy takes the form of heat energy.

An important point to observe in connection with equation (10) is that the carbon and oxygen atoms have not changed in form; they have merely combined to form a molecule of CO_2. The amount of heat energy per molecule actually released is relatively very small compared with that which is released during an atomic nuclear transformation wherein completely different atomic structures are produced during the reaction. But the fact that the CO_2 chemical reaction is self-sustaining makes the conversion from chemical (or binding) energy to heat energy a simple one; and one upon which a large part of our national economy depends.

One pound of coal (which possesses a calorific value of 13,650 Btu/lb) will develop 4 kwhr of heat energy. In the ordinary conversion process to steam, then to mechanical energy, and finally to energy in electrical form, about 3 kwhr (of the intrinsic calorific value of 4 kwhr) is lost, leaving 1 kwhr of electrical energy generated per pound of coal burned.

13. The Equivalence of Mass and Energy. In 1905, Einstein stated that mass and energy were equivalent because his study of relativity clearly indicated that the inertial mass of a body increased as its velocity increased.[7] He concluded that the amount of energy W which would transform into an equivalent mass m (or vice versa) was given by the equation

$$W = mc^2 \tag{11}$$

[7] This increase in mass is not significantly large for velocities which are equal to or less than one-fifth the velocity of light, since at one-fifth the velocity of light the increase in mass is only about 2 per cent.

where c is the velocity of light and all units are expressed in primary units of the adopted system.

Reference to page 4, will show that energy is dimensionally equal to $m^1l^2t^{-2}$ which is, of course, equal to mv^2, but one could not infer from this dimensional equivalence that the actual physical transformation postulated by Einstein would occur in nature. It does, however, occur, and the law of the conservation of energy now implies that *energy plus mass can neither be created nor destroyed but only altered in form.*

Example. Let it be required to find the amount of energy theoretically generated by the complete disintegration or annihilation of 1 kg of mass.

Employing mks units, say, in connection with equation (11) one finds that the energy equivalent of 1 kg of mass is

$$W = 1 \times (3 \times 10^8)^2 = 9 \times 10^{16} \quad \text{joule or watt-sec}$$

or

$$W_{\text{kwhr}} = \frac{9 \times 10^{16}}{3.6 \times 10^6} = 25 \times 10^9 \quad \text{or} \quad 25 \text{ billion kwhr}$$

Since our national consumption is about 300 billion kwhr annually, the fantastic result obtained (25 billion kwhr) represents about $\frac{1}{12}$ of our annual consumption of electrical energy. No one has figured out as yet just how to get 25 billion kwhr from 1 kg (2.2 lb) of coal, however.

The equivalence of mass and energy appeared quite largely as a mathematical fiction until recently when certain nuclear reactions demonstrated forcibly that slight changes in atomic structures (which were accompanied by a loss of mass) could unleash great quantities of heat and radiant energy.

PROBLEMS

1. Given the three electrical charges located at vertices of the right triangle shown in Fig. 12. Check the value of the resultant force, $f_R = 278$ newtons, and find the value of the angle θ in degrees.

2. Find the resultant force on charge Q_2 of Fig. 3, page 30. Work in rationalized mks units, specifying the direction of the resultant force relative to the direction of the line joining Q_1 and Q_2.

3. What is the *gravitational* force of attraction (in dynes) between two electrically neutral particles having masses of 9.1×10^{-28} g and 1.67×10^{-24} g respectively if the distance of separation between their centers of mass is 10^{-4} cm?

Note: Newton's law of gravitational attraction reads

$$f = \frac{m_1 m_2}{k_0 r^2}$$

where k_0, the gravitational permittivity, is equal to 1.5×10^7 in cgs units.

4. What is the *electrical* force of attraction between the two particles described in Prob. 3 if each possesses a charge of 1.6×10^{-19} coulomb, one positive and the other

negative? A free-space medium is assumed. Compare this force with that obtained in Prob. 3.

Note: Work in *unrationalized* stat-cgs units where Coulomb's inverse square law reads

$$f = \frac{Q_1 Q_2}{\epsilon_0' r^2}$$

The value of ϵ_0' in unrationalized stat-cgs units is equal to 1.

5. What is the magnitude of f_R in Fig. 12 if the entire system of charges is immersed in distilled water, the relative permittivity of which is 80?

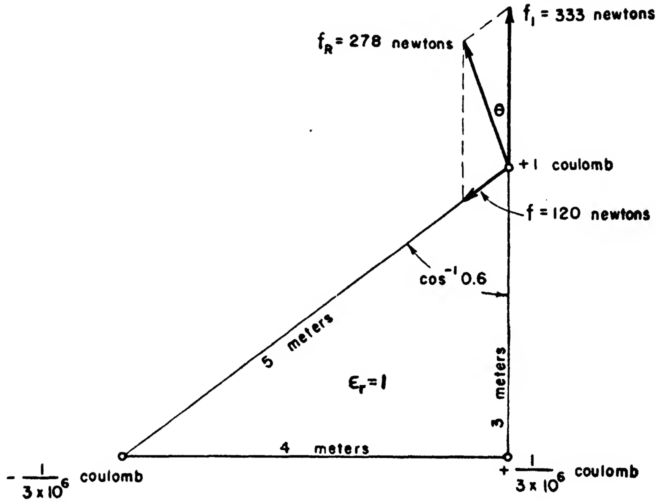

FIG. 12. See Prob. 1.

6. Two small bodies, one possessing twice as many net electrons as the other, are separated from one another in air by a distance of 2 m, and they experience mutual forces of repulsion of 0.0002 newton. What is the number of net electronic charges on each of the bodies?

7. A man loading a conveyor belt lifts 100-lb loads 4 ft (against the force of gravity) at the average rate of 2 loads per minute. How many kilowatt-hours of energy (or work) does he accomplish in an 8-hr day? *Ans.* 0.1446 kwhr.

8. If the job described in Prob. 7 can be done electrically with electrical energy which costs 4 cents per kilowatt-hour, what would be the cost of the electrical energy assuming that the electrical system has an efficiency of 0.8?

Note:

$$\text{efficiency} = 0.8 = \frac{\text{useful energy output}}{\text{total energy input}}$$

9. What number of kilowatt-hours can theoretically be generated by 1,000,000 cu ft of water falling through a distance of 100 ft? The weight of a cubic foot of water is about 62.5 lb.

10. Assume that the energy expended under a particular set of conditions (as, for example, the operation of a 200-watt incandescent lamp) can be written as a function of time in equational form:

$$W = 200t \quad \text{joules or watt-sec}$$

where t is time expressed in seconds. What is the amount of energy expended over a 1-hr period in kilowatt-hours?

11. The force on an electrical charge of 0.1 coulomb (due presumably to other charged bodies in the vicinity) may be written equationally in mks units as

$$f = 1000l_m \quad \text{newtons}$$

where l_m is the linear displacement along the path of motion expressed in meters. How many joules (or newton-meters) of energy are expended by an outside agent (or worker) in moving the charge from $l = 2$ cm to $l = 5$ cm? It is assumed that the worker opposes directly the electrical force at all points along the path of motion.

12. What is the electrical charge of (*a*) the nucleus of a magnesium atom, (*b*) the nucleus of an iron atom, expressed in coulombs? (Refer to Appendix C.)

13. After referring to Appendix C, draw a schematic representation of a beryllium atom. The mass number of beryllium is 9.

14. What is the frequency of a radio wave if its wave length is 200 m?

15. One of the wave lengths of light radiated by an excited neon atom is the result of an electron changing from an energy level of 18.90 electron volts (30.24×10^{-19} joule) to an energy level of 16.80 electron volts (26.88×10^{-19} joule). What is the frequency of the radiated light waves in cycles per second? What is the wave length?

16. What is the color of light if its frequency is 4.25×10^{14} cycles/sec?

17. An ultra-violet radiation results from an electron (of an excited atom) changing from an energy level of 16.8 electron-volts (above normal) to its normal energy level. What is the number of waves per centimeter of the radiated energy?

18. What is the *number of* electron volts (of energy) per gram-calorie (of energy)?

19. The heat energy released in the formation of CO_2 is 94,000 g-cal per 6.03×10^{23} molecules of CO_2. What is the heat energy released per molecule expressed in electron volts?

20. A small generating station burns 2720 lb of crude oil to generate 2710 kwhr of useful electrical energy. If each pound of oil releases 17,000 Btu of heat energy, what is the overall efficiency of operation?

Note: As a ratio

$$\frac{\text{kwhr}}{\text{Btu}} = 2.93 \times 10^{-4}$$

therefore

$$(\text{No. of})\ \text{Btu} \times (2.93 \times 10^{-4}) = \text{kwhr}$$

Overall efficiency is defined as

$$\frac{\text{useful energy output}}{\text{total energy input}}$$

21. If the mass of a helium nucleus could be transformed entirely into energy, what number of joules of energy would be obtained from this transformation? (Two-significant-figure accuracy is sufficient here.)

22. If, in a wholly hypothetical case, an electron should collide with a proton in such a manner that both masses were completely annihilated, what number of joules of energy would be generated? (Two-significant-figure accuracy is sufficient here.)

23. A classical nuclear reaction is shown schematically in Fig. 13. The problem is to calculate (from the specified data) the kinetic energy and the velocity of each of the helium nuclei, He^4, employing the law of conservation of mass and energy. The

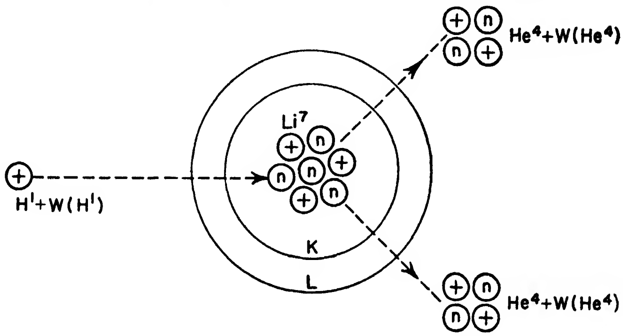

FIG. 13. Collision of hydrogen nucleus (H^1) and lithium nucleus (Li^7) produces two helium nuclei (He^4). $W(H^1)$ and $W(He^4)$ represent kinetic energies.

precise mass data are taken from the reference cited on page 35, and the superscripts employed refer only to the number of massive particles in the nuclei.

(*a*) H^1 (a hydrogen nucleus) initially possesses a kinetic energy of $W(H^1) = 1.6 \times 10^{-14}$ joule.

(*b*) The initial mass $(Li^7 + H^1) = 13.322 \times 10^{-27}$ kg.

(*c*) The final mass $(2He^4) = 13.292 \times 10^{-27}$ kg.

Note: The nuclear reaction may be written as a *mass* equation:

$$Li^7 + H^1 + \frac{W(H^1)}{c^2} = 2He^4 + 2\frac{W(He^4)}{c^2}$$

where $W(He^4)$ is the kinetic energy of each helium nucleus. It will be assumed here that the mass of each helium nucleus, He, remains constant at its specified value irrespective of the velocity of the particle.

CHAPTER III

Voltage, Current, and Power

An agency like electricity which is used principally to change energy from one form to another must be capable of possessing energy. In measuring the potential energy of a unit charge, the concept of *potential* is employed. Electrical potential may, at first, appear to be an abstract

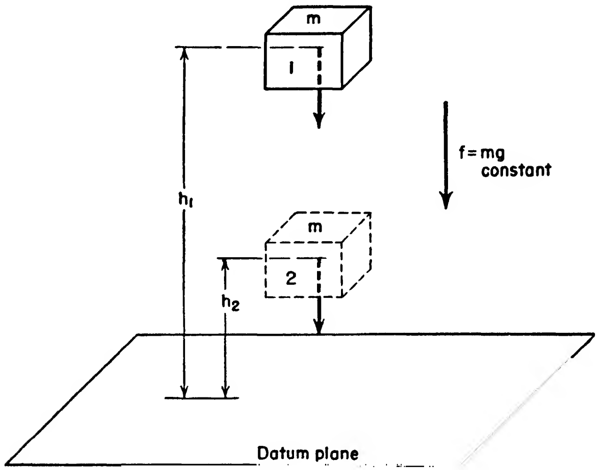

FIG. 1. Illustrating gravitational potential.

concept, and for this reason a brief discussion of gravitational potential will be given in order that the reader may later appreciate the analogy which exists between gravitational potential and electrical potential.

1. Gravitational Potential. If the datum plane shown in Fig. 1 is the earth's surface, we might *define* the gravitational *potential* of any point above the plane as

$$V_g = \frac{\text{potential energy of mass } m \text{ at any point}}{m} = \frac{W}{m} \tag{1}$$

In so doing, V_g, the gravitational potential, is defined in such a manner that it is potential energy per unit mass; dimensionally equal to l^2t^{-2}.

Moreover, we have defined the zero potential plane as being at the earth's surface.

The gravitational potential of a point characterizes that point in so far as it tells precisely the potential energy that a unit mass would possess if it occupied that position. For the simple case shown in Fig. 1, the concept of gravitational potential is somewhat elaborate because height h above the datum plane tells essentially the same thing as does V_g.

As applied to position 1 of Fig. 1, the potential above the zero potential datum plane is

$$V_{g1} = \frac{W_1}{m} = \frac{mgh_1}{m} = gh_1 \tag{1-a}$$

where g is the gravitational acceleration, 9.81m/sec/sec in mks units.

By means of a simple extension of the definition given in equation (1), the *potential difference* between points 1 and 2 is

$$V_{12} = V_{g1} - V_{g2} = \frac{W_1 - W_2}{m} = g(h_1 - h_2) \tag{1-b}$$

If for some reason we should choose by definition the gh_2 potential level as our *reference level* and it is understood by all concerned that the gh_2 level is the reference level, we could speak of positive potentials and of negative potentials and the meanings would be clear. Potential differences between any two points could also be determined. The potential difference between position 1 and the datum plane, for example, might be determined as

$$V_{10} = g(h_1 - h_2) - g(-h_2) = gh_1$$

gh_2 being employed as the reference level.

2. Zero Electrical Potential. An electrical charge in the vicinity of other charges is capable of doing work in much the same manner that mass m of Fig. 1 is capable of doing work by virtue of being in the vicinity of the earth's mass. In the case of gravitational potential, the zero potential plane is ordinarily taken to correspond to the earth's surface because a mass m at this level has ordinarily lost its ability to do work. With this choice of zero potential plane, one can readily visualize levels below the earth's surface as negative potential regions. But again it is pointed out that *the reference level we arbitrarily choose as zero is a matter of definition,*[1] like that given in equation (1).

[1] If, for example, we had *defined* the gravitational potential of the points above the earth's surface as *the work done (by an outside agent) upon a unit mass in bringing this mass from an infinitely great distance away from the earth to points above the earth's*

In a strict manner of speaking, we define a zero electrical potential region as a region where an electrical test charge (or exploring charge) is incapable of doing any work. This region might be

(*a*) Any region which is so far removed from other electrified bodies that the test charge experiences zero electrical force.

(*b*) A region in the vicinity of a set of electrical charges where, due to forces of both attraction and repulsion, the resultant force on the test charge is zero.

(*c*) Moist earth, since the earth is electrically neutral and so large that any man-made charge will not appreciably affect this neutrality.

(*d*) A large metal plate or screen which serves as a *ground* (to the minute distributions of charge employed in radio and ultra-high-frequency systems) in much the same manner that moist earth serves as a *ground* to larger man-made charge distributions.

Potentials measured from *true zero potential levels* are called *absolute* potentials.

We may, however, select any absolute potential level as the zero or reference level from which to measure electrical potential. In general, we are concerned only with potential differences and, where this is the case, the selection of the zero or reference level is *wholly a matter of choice*. The selection is usually made with a view toward simplifying the complete specifications as, for example, in a non-grounded circuit where we arbitrarily select one wire or bus as the zero potential region and reckon all other potentials relative to this bus. By so doing, a single number (of units of potential) placed alongside any point in the circuit specifies the potential difference between the potential of this point and that of the arbitrarily selected zero potential bus.

3. Electrical Potential. We *define* the *electrical potential* of a point in space (or in a circuit) as

$$E \text{ (or } V) = \frac{\text{potential energy of charge } Q \text{ at the point}}{Q} = \frac{W}{Q} \tag{2}$$

where it is understood that the potential energy may be measured from any arbitrarily selected reference level, as was the gravitational potential energy in equation (1).

Although electrical potential is defined in such a manner that it is numerically equal to energy per unit charge it is not energy. Whereas

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surface, the zero potential region or level would have been at $h = \infty$ in Fig. 1. In this case all the potentials involved in the example considered would have been negative values, with the greatest negative value appearing at the earth's surface. Since this definition leads to unnecessary complications, it is avoided simply by selecting (by definition) a more suitable zero potential plane from which to measure potential.

energy is dimensionally $m^1l^2t^{-2}$, electrical potential is dimensionally $m^1l^2t^{-2}Q^{-1}$.

If, in connection with equation (2), we choose a *zero* potential *energy* plane as reference, then

$$E = E_{abs} = \frac{\text{potential energy of } Q \text{ relative to zero energy}}{Q} = \frac{W_{abs}}{Q} \qquad (2\text{–}a)$$

The *potential difference* between two points in space (or in a circuit), say points 1 and 2, is

$$E_{12} = E_{abs1} - E_{abs2} = \frac{W_{abs1} - W_{abs2}}{Q} \qquad (3)$$

and it is plain from this equation that, in so far as the value of E_{12} is concerned, E_{12} might just as well have been calculated from

$$E_{12} = \frac{(W_{abs1} + W_c) - (W_{abs2} + W_c)}{Q} \qquad (3\text{–a})$$

where W_c is any arbitrary value of potential energy we choose to select. If, for example, we choose to have the potential of point 2 as our reference level, we simply write

$$E_{12} = \frac{W_1 - 0}{Q} \qquad (3\text{–b})$$

which is numerically the same as equation (3–a) if we set $W_c = -W_{abs2}$, thereby making $W_1 = W_{abs1} - W_{abs2}$, the potential energy difference between points 1 and 2.

The term electrical potential is usually abbreviated to simply *potential*, and the terms *potential* and *voltage* are used synonymously in engineering literature.

Units. From equation (2), it is plain that the *number of* mks units of potential (or voltage) would be found from the ratio joules/coulomb; and in like manner any ratio expressing (units of energy)/(unit charge) would yield units of potential.

For convenience, joules/coulomb are called volts. The *volt* is the primary unit of potential or voltage in the mks system, and it is the unit ordinarily employed in engineering practice.[2]

4. Voltage Rise and Voltage Drop. If a positive charge is moved from one point to another in a circuit (or in space), it necessarily follows

[2] Two other units of potential are sometimes encountered in theoretical work, the statvolt and the abvolt. The statvolt is a unit which is larger than the volt by the factor 300; whereas the abvolt is a unit which is smaller than the volt by the factor 10^8.

that the starting point is less positive and the terminating point is more positive than before the transfer of charge took place. To move a positive charge from one point to another in a region which is initially electrically neutral, for example, will render the starting point negative and the terminating point positive. The charge is said to move in the − to + direction. A transfer of this kind, of course, requires energy from some outside source, because work has to be done to move the charge against the force of attraction which the positive charge has for the negative region it created. Since the positive charge in its new position is capable of doing work (in returning to its starting place) it is said to have been raised in potential as it moved in the − to + direction.

More specifically, if *1 positive coulomb* of charge (in moving in the − to + direction) *absorbs 1 joule* of energy (from some outside source) it has experienced a *rise* of *1 volt* in potential.[3]

It follows from the law of the conservation of energy that a charge which has been raised in potential can in turn *deliver* or *release* energy upon returning to its starting point. The precise form in which this release energy will appear depends upon the nature of the return circuit. This return circuit may be so fashioned that the delivered energy will appear as heat, light, mechanical or chemical energy.

If *1 positive coulomb* of charge *releases 1 joule* of energy (in moving in the + to − direction) *it falls 1 volt* in potential.

With proper rearrangement and interpretation of equation (2), we obtain two of the most fundamental equations of electrical engineering, namely,

$$\text{Generated energy} = EQ \quad \text{(where } Q \text{ acquires energy)} \tag{4}$$

$$\text{Released energy} = VQ \quad \text{(where } Q \text{ releases energy)} \tag{5}$$

In this connection, E is the generated voltage rise between two points as, for example, between the − terminal and the + terminal of an electrical generator. (See Fig. 2.) V is the dissipated voltage drop between two points as, for example, the terminals of the device which receives electrical energy and converts this energy into the desired form.

[3] It will be shown later that in metallic circuits only electrons or (negative charges) actually move, whereas in electrolytic circuits and in gas tubes both positive and negative charges move. In order to avoid confusion, we normally deal only with the *positive charge equivalent* of all the charge movements that are present in a given situation. It should be plain that negative charge in going from − to + will do precisely what a positive charge does in going from + to −. If, then, by convention, we employ only the *positive charge equivalent* of all charge movements, we can indicate polarities on a circuit diagram without the uncertainties which would inevitably follow if both types of charge movements were depicted on the same diagram.

Example. An electric motor is a device for transforming electrical energy into mechanical energy. For the case shown in Fig. 2, the motor delivers 14,752 lb-ft of mechanical work as a result of +100 coulombs passing in the + to − direction through the motor windings.

Let it be required to find the potential difference between the terminals of the motor.

In order to use equation (5) in mks units, we convert the pound-feet of released electrical energy to joules:

$$(\text{No. of) joules} = (\text{No. of}) \cancel{\text{lb-ft}} \times \frac{\text{joules}}{\cancel{\text{lb-ft}}} = 14{,}752 \times 1.356 = 20{,}000$$

Then

$$V = \frac{\text{joules}}{\text{coulombs}} = \frac{20{,}000}{100} = 200 \text{ volts} \quad \text{(potential difference)}$$

In saying that the energy released by the electrical charge all appears at the shaft of the motor, we have neglected certain small energy losses within the

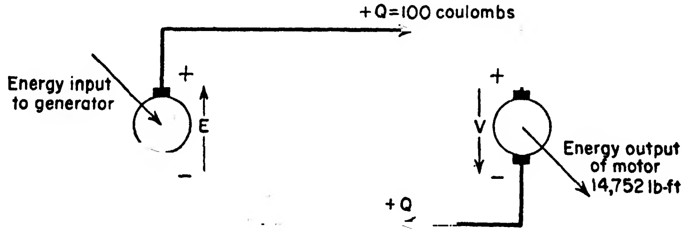


Fig. 2. Elementary example of energy transformation.

motor like the heat energy which appears in both the copper windings and in the bearings. These heat losses are relatively small in a good motor and are incidental to the transformation of energy which is desired, namely, the transformation from electrical energy to mechanical energy.

5. Sources of Voltage Rise. Until practical sources of potential difference like batteries, generators and thermocouples have been described, the reader is asked to accept the fact that these devices function in a manner to maintain potential differences across their terminals. As such they function like electric pumps which circulate positive charge in a direction from − to + through the pump when the terminals are closed through an external dissipative circuit which permits the circulation of charge.

Sources of this kind are called sources of electromotive force, abbreviated emf.

6. Current (I). The time rate of change of electrical charge across a specified cross-sectional area is called the electrical current through

this area. As applied to an ordinary circuit, the movement of charge is confined to well-defined paths by the conductors used to make up the circuit, and we speak of the current *in* the circuit. Unless the current density within the conductors themselves is of importance, the actual cross-sectional area of the conductors is of little theoretical importance.

The current in a circuit is defined as

$$I = \frac{Q}{t} \quad \text{or} \quad i = \frac{dq}{dt} \tag{6}$$

where the lower-case letters indicate that, in general, q (and hence i) may vary from instant to instant, in which case instantaneous values of charge and current must be considered.

For the present, it will be assumed that the charge passes a specified point (or cross-sectional area) of the circuit in one direction at a uniform rate. In this case, equation (6) becomes

$$I = \frac{\Delta Q}{\Delta T} = \frac{Q}{T} \tag{7}$$

where the same increment of charge (ΔQ) passes a specified point (at which the current is defined) during each increment of time, ΔT. Thus the symbol I represents a steady current (one which does not vary with time) and as such is called a direct current. The symbol i is usually employed to denote a time-varying current.

Units. The customary unit of current employed in practice is the coulomb per second which for convenience is called an *ampere*.[4] The ampere is the primary unit of current in the mks system.

Where small values of current are involved, secondary units like the milliampere and the microampere are sometimes employed in specifying the magnitude of a particular current. A milliampere is $1/1000$ of an ampere, and a microampere is $1/10^6$ of an ampere. (See page 9.) It should be recognized that in general secondary units of this kind cannot be used in theoretically derived equations unless appropriate numerical coefficients are included to account for the secondary units.

Example. If the 100 coulombs referred to in Fig. 2 pass any point in the series circuit during a 5-sec interval, the current in the series circuit is 100/5 or 20 amp.

[4] Another unit of current which is sometimes encountered in theoretical derivations is the *abampere*. The abampere is a unit of current which is ten times larger than the ampere.

7. Power (*P*). Since the *power* involved in any energy transformation is *the time rate of energy transformation,* electric power is simply

$$P = \frac{W}{T} \quad \text{or} \quad p = \frac{dw}{dt} \tag{8}$$

where the lower-case letters indicate that, in general, w (and hence p) might vary from instant to instant, in which case instantaneous values of energy transformation and power would have to be considered.

For the present, it will be assumed that the same increment of energy transformation (ΔW) takes place during each increment of time (ΔT) which is being considered. In this case the power is simply

$$P = \frac{\Delta W}{\Delta T} = \frac{W}{T} \tag{9}$$

where W is the energy transformed in time T. It follows from equations (4) and (5) that

$$P_{\text{gen.}} = \text{generated power} = \frac{E\,\Delta Q}{\Delta T} = EI \tag{10}$$

$$P_{\text{rel.}} = \text{released power} = \frac{V\,\Delta Q}{\Delta T} = VI \tag{11}$$

where, if E (or V) is in volts and I is in amperes, P is given in watts. For $P_{\text{gen.}}$ to be positive, $+I$ must flow in the $+E$ direction ($-$ to $+$ direction).
For $P_{\text{rel.}}$ to be positive, $+I$ must flow in the $+V$ direction ($+$ to $-$ direction).

Units. The watt represents a time rate of energy transformation of 1 joule/sec. It is the primary unit of power in the mks system of units. A secondary unit of power which is widely used in practice is the kilowatt, abbreviated kw. The prefix *kilo* indicates that the kilowatt is equivalent to 1000 watts.

One watt of power developed over a period of 1 sec represents 1 joule of energy. For this reason a joule is sometimes referred to as a watt-second. One killowatt of power developed over a period of 1 hr represents 1 kwhr of energy.

Example. The energy transformation indicated in Fig. 2 amounts to 20,000 joules. If this transformation takes place in 5 sec, the rate of energy transformation or power is 20,000/5 joules/sec or 4000 watts. This same result might have been determined from the product of the voltage (200 v) and the current (20 amp) in accordance with equation (11).

8. Energy and Power in Electrical Circuits. In the circuit arrangement shown in Fig. 3, several important relationships are deducible from equations (4) and (5). The voltage rise from e to a through the generator (the source of emf) is symbolized as E_{ea}, and the voltage drops across the various dissipative elements are symbolized by V's with appropriate subscripts. The + and − symbols used in connection with the circuit elements of Fig. 3 indicate the higher and lower potential terminals respectively of the circuit elements.

If the generator raises Q_{ea} coulombs from − to + in time T, it supplies the electrical circuit with $E_{ea}Q_{ea}$ joules of energy (in time T) or with $E_{ea}I_{ea}$ watts of power throughout this period. At least $E_{ea}Q_{ea}$ joules

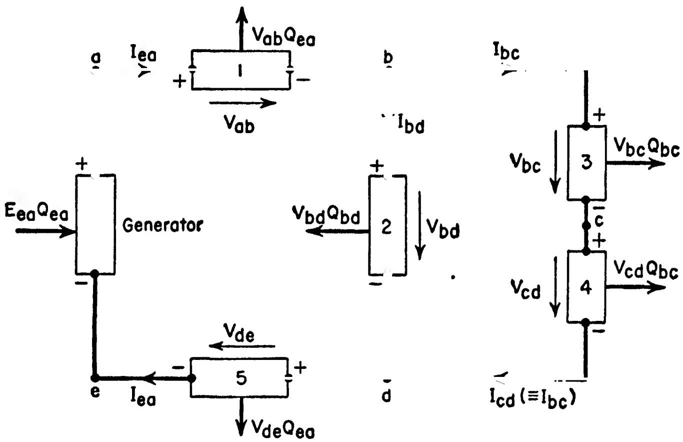


FIG. 3. Electrical network.

of energy must be used to drive the generator, and this is represented by the $E_{ea}Q_{ea}$ arrow entering the generator, presumably from some prime mover.

Since Q_{ea} ($= I_{ea}T$) coulombs of charge drop through a potential difference of V_{ab} in passing through dissipative element 1, $V_{ab}Q_{ea}$ joules of energy are converted from electrical form to some other form, and hence this amount of energy is shown as leaving the electrical circuit by the $V_{ab}Q_{ea}$ arrow. ($Q_{ab} = Q_{ea}$.) The energy leaving the circuit at element 1 would, for example, take the form of heat and light if element 1 is an ordinary incandescent lamp; take the form of mechanical energy if the element is an electric motor; or take the form of chemical energy if the element is a storage battery undergoing a charging process.

Similar explanations might be given for the other arrows pointing out from the electrical network.

Applying the principle of the conservation of energy to the network shown in Fig. 3, one may write

$$\underset{\text{(energy transformed by dissipative elements)}}{(V_{ab} + V_{de})Q_{ea} + V_{bd}Q_{bd} + (V_{bc} + V_{cd})Q_{bc}} = \underset{= \text{(energy supplied)}}{E_{ea}Q_{ea}} \tag{12}$$

A corresponding power equation may be obtained by dividing equation (12) through by time T. Thus

$$\underset{\text{(power transformed by dissipative elements)}}{(V_{ab} + V_{de})I_{ea} + V_{bd}I_{bd} + (V_{bc} + V_{cd})I_{bc}} = \underset{= \text{(power supplied)}}{E_{ea}I_{ea}} \tag{13}$$

The potential difference between any two points is the same regardless of the path traversed in going from one point to the other. Hence the voltage drop across element 2 (of Fig. 3) is the same as the voltage drop across the series combination of elements 3 and 4. We may write this physical fact in the form of an equation as

$$V_{bd} = V_{bc} + V_{cd} \tag{14}$$

where the subscripts which are employed indicate the points in the circuit which are being considered.

Since electric charge does not accumulate at a point (or cross-sectional area) as a result of current flow, the current leaving junction b in Fig. 3 must equal the current coming into this junction. Thus

$$I_{bd} + I_{bc} = I_{ea} \tag{15}$$

With the aid of the relationships stated in equations (14) and (15), equation (13) may be written in the form

$$(V_{ab} + V_{bd} + V_{de})I_{ea} = E_{ea}I_{ea} \tag{16}$$

Equation (16) indicates that the sum of the voltage drops around the closed loop *abdea* in Fig. 3, $(V_{ab} + V_{bd} + V_{de})$, is equal to the voltage rise in this loop, E_{ea}; a fact which will be elaborated upon in the following section.

9. Kirchhoff's Laws. The two guiding principles of circuit analysis are

(1) *Kirchhoff's Current Law.* The sum of the currents directed *away* from a junction is equal to the sum of the currents directed *toward* the junction.

(2) *Kirchhoff's Emf Law.* The sum of the voltage drops around any closed loop of a network equals the sum of the voltage rises around this loop.

The current law follows directly from the fact that electric charge does not accumulate at a point (or cross-sectional area) as a result of

current flow. An example of the application of this law to a circuit junction is given in equation (15).

The emf or voltage law follows directly from the fact that the potential of any point in the circuit is a fixed or single-valued quantity at any instant of time. In tracing completely around any closed loop (starting at any point and ending at this same point) as much potential *drop* will be encountered as potential *rise*; otherwise the starting side of the point would be at a difference of potential with respect to the terminating side. A potential or voltage *drop* is encountered in going from + to −, and a voltage *rise* is encountered in going from − to +.

The voltage law as applied to any particular loop in a network may be written in equational form in several different ways, all of which are equivalent. In the interest of systematizing the simultaneous voltage equations which will be encountered later, it is suggested that voltage equations be written in the following manner:

(1) Select the − to + direction through the emf source as the positive tracing direction around the loop. (If more than one emf source is present in the loop, use the one having the greatest voltage rise in deciding upon the positive tracing direction.)

(2) Write down the algebraic sum of the potential differences which exist across the dissipative elements of the loop; these potential differences being classed as positive if the positive tracing direction traces them in the + to − direction, negative if the positive tracing direction traces them in the − to + direction.

(3) Equate the algebraic sum of the above dissipative voltages to the algebraic sum of the emf's which are encountered in tracing the loop; an emf being classed as positive if the positive tracing direction traces the emf from − to +, negative if the positive tracing direction traces the emf from + to −.

Application of the voltage law to loop *eabde* of Fig. 3 yields

$$V_{ab} + V_{bd} + V_{de} = E_{ea} \tag{17}$$

since voltage drops (+ to −) are encountered in going from a to b, from b to d, and from d to e. A voltage rise (− to +) is encountered in going from e to a.

Application of the voltage law to loop *bcdb* yields

$$V_{bc} + V_{cd} - V_{db} = 0 \tag{18}$$

In this case the right-hand side of the equation is zero since no emf source exists in the *bcdb* loop. The potential difference across element 2 (V_{db}) is classed as negative on the left-hand side of equation (18) since

the positive tracing direction traces this potential difference in the − to + direction.

Application of the voltage law to loop *abcdea* of Fig. 3 yields

$$V_{ab} + V_{bc} + V_{cd} + V_{de} = E_{ea} \tag{19}$$

Example. The network shown in Fig. 4 contains two generators, *A* and *B*, and dissipative elements 1, 2, and 3. The information known about the network is shown on the diagram and it is required to find

$$E_A = E_{ea} \qquad I_B = I_{eb} \qquad V_{bc}$$

Application of the current law to junction *b* indicates that

$$I_{ab} + I_{eb} = I_{bc} \quad \text{from which} \quad I_{eb} = 6 - 2 = 4 \quad \text{amp}$$

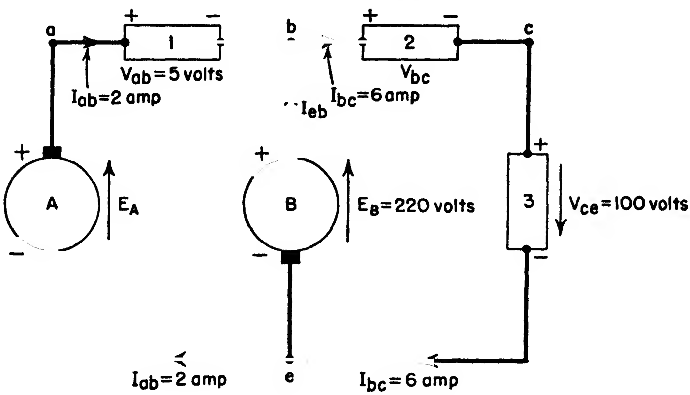


FIG. 4. See Example.

Application of the voltage law to loop *eabe* yields

$$V_{ab} = E_A - E_B \quad \text{from which} \quad E_A = 5 + 220 = 225 \quad \text{volts}$$

If the voltage law is applied to loop *ebce*, it is found that

$$V_{bc} + V_{ce} = E_B \quad \text{from which} \quad V_{bc} = 220 - 100 = 120 \quad \text{volts}$$

10. Circuit Directions. Problems like the one given above occur frequently in engineering practice. They can be solved very simply if the proper distinction is made between voltage rises and voltage drops. After a little practice, the double subscript notation used in Figs. 3 and 4 may be dispensed with, and singly subscripted currents with an assigned arrow direction can be used in most cases advantageously. The arrow direction of the current then defines the polarity of the *voltage drop* through dissipative elements *in the arrow direction* as being from + to −.

The polarities of the generators must be known and, if the current flows from − to + through a generator, the generator is delivering *EQ*

energy or EI power to the electrical system. In case the current is actually flowing from + to − through a generator (or battery), the device is actually receiving power from the electrical system to the extent of EI. This power must of course come from some other generator in the system.

Example 1. In the case shown in Fig. 4, let $I_{ab} = I_A$ and $I_{eb} = I_B$. The power delivered to the system by generators A and B are

$$P_A = E_A I_A = 225 \times 2 = 450 \quad \text{watts}$$

$$P_B = E_B I_B = 220 \times 4 = 880 \quad \text{watts}$$

The total dissipative power is

$$V_1 I_A + V_2(I_A + I_B) + V_3(I_A + I_B) = (5 \times 2) + (120 \times 6) + (100 \times 6) = 1330 \quad \text{watts}$$

If a 1-hr period of time is considered, the energy delivered to the system by both generators is

$$(P_A + P_B)T = 1330 \times 3600 = 4{,}788{,}000 \quad \text{watt-sec or joules} = 1.33 \quad \text{kwhr}$$

Example 2. In the circuit shown in Fig. 5 E_A is 8 volts and E_B is 6 volts with polarities of the voltage rises as specified by the − and + signs. Obviously, there is a single branch current, I, in this case if switch S is open and this current flows in the indicated arrow direction since $E_A > E_B$. It is therefore natural to choose the *clockwise* (or $+I$) direction around the current-carrying loop as the *positive tracing direction* when equations (10) and (11) are used to find the power distribution of the system.

Applying equation (10) to generator A, we find

$$P_{\text{gen.}} = E_A I = 8 \times 10 = 80 \text{ watts} \quad \text{(power delivered to the system)}$$

It will be observed that both E_A and I are entered into the above equation as positive quantities because the $+E$ (or − to +) direction and the $+I$ direction both agree with our arbitrarily selected clockwise tracing direction.

Applying Kirchhoff's emf law to the loop, we find

$$V_{ab} = E_A - E_B \quad \text{or} \quad V_{ab} = 8 - 6 = 2 \quad \text{volts}$$

Applying equation (11) to the dissipative element ab, we find

$$P_{\text{rel.}} = V_{ab} I = 2 \times 10 = 20 \text{ watts} \quad \text{(power absorbed by element } ab\text{)}$$

Since both the positive direction of V_{ab}(+ to −) and the direction of $+I$ coincide with the positive tracing direction, they are entered into the equation as positive quantities, and the result is positive released power as, of course, it must be if element ab is not a generator.

Applying equation (10) to generator B, we find

$$P_{\text{gen.}} = -E_B I = -6 \times 10 = -60 \text{ watts} \quad \text{(power generated by generator } B\text{)}$$

Since the positive tracing direction (clockwise in this case) is opposed to the positive direction of voltage rise, a minus sign is used in connection with E_B and the result is a *negative generated* power. This *negative generated power*

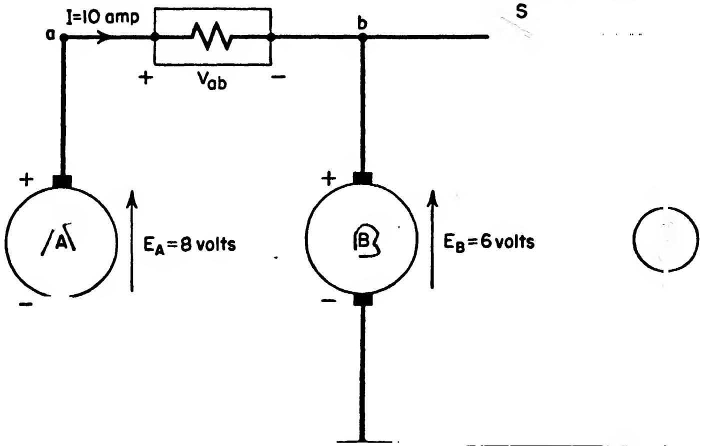


FIG. 5. See Example 2.

represents electrical power which is *absorbed* by generator B since generator B is functioning as a *negative* generator. If B is an electromagnetic generator it is necessarily connected mechanically to a prime mover and the power *absorbed* by generator B is delivered to this prime mover in the form of mechanical power. If B is a storage battery, it undergoes a charging process in which the power absorbed by B effects chemical changes at the plates of the battery.

The power distribution of the system may be written as

$$\begin{aligned} P_{\text{gen.}} &= P_{A\text{ gen.}} + P_{B\text{ gen.}} = P_{\text{rel.}} \\ &= 80 \quad + (-60) = 20 \quad \text{watts} \end{aligned}$$

11. Electrical System of an Automobile. The electrical system of an automobile, considerably simplified for our purposes, is shown in Fig. 6. The diagram is more or less self-explanatory, G being the generator, B the storage battery, S the starter, L the lights, and D the distributor. The necessary make-and-break mechanism for the low-voltage winding of the ignition coil is not shown.

The reason for *grounding* the positive terminal of the battery (and hence of the generator) is that the positive terminal corrodes the connecting wire somewhat more than does the negative terminal. Since the ground strap is shorter and easier to replace than the cable connection to the starter, the replacement cost of the ground strap is less. It will be observed that all the points of the low-voltage system are

either at chassis potential or lower (that is, negative relative to the chassis) if the + terminal of the battery is connected to the chassis as shown in Fig. 6. The potential difference distributed to the spark plugs is relatively great in magnitude (in the order of 10,000 volts) and is of an oscillatory nature. This potential difference is of no importance to the present discussion, which is concerned only with the low-voltage d-c system of the automobile.

The starter S is an electrical motor which, when the starter switch is closed, takes a relatively large current from the battery. The connecting cables from the battery to the starter must be of large cross-sectional area in order to reduce to practical limits the heat losses that occur in

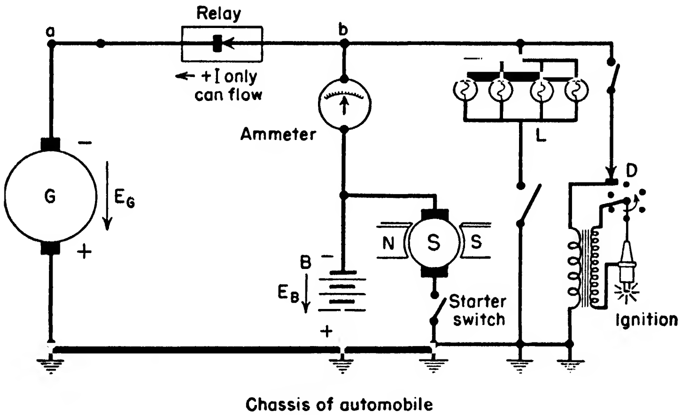


FIG. 6. Electrical system of an automobile (simplified).

all current-carrying wires. As the following example will show, the magnitude of the starter current is in the order of 100 amp on the average but it may in certain cases be as great as 400 or 500 amp.

It will be observed from the diagram shown in Fig. 6 that the starter current does not flow through the ammeter, which is employed primarily to inform the operator whether the battery is charging or discharging during normal operation. If current ($+I$) flows in the $+E_B$ direction (that is, from − to + through the battery) the battery is supplying energy to the system and the ammeter reads *discharge*. If, however, E_G is greater in magnitude than E_B the battery receives $+I$ in the $-E_B$ direction (that is, from + to −) and the ammeter reads *charge*. A reverse-current relay is placed in the generator-battery circuit so that the battery cannot deliver current to the generator for obvious reasons. Although the details of this relay are not shown in Fig. 6, the operation of the relay is such that it functions as a switch to keep the

generator-battery circuit *open* unless E_G is greater in magnitude than E_B. When E_G is greater in magnitude than E_B, $+I$ can flow in the counterclockwise direction around the generator-battery loop as indicated, and in this case the battery undergoes a charging process and the generator supplies power to the lights and ignition system.

Although the above description of the circuit arrangement shown in Fig. 6 is wholly qualitative in nature, it nevertheless is representative of one very important phase of an electrical engineer's daily work. That is: (*a*) given a circuit diagram of any electrical system, (*b*) find the power distribution (qualitatively) from a careful study of the potential distribution. Or, more concisely, given the circuit diagram, find how the system *works*.

Example. Let it be required to estimate the magnitude of the starting current in Fig. 6 from the following assumed data:

1. The starter power output is 420 watts. (See estimate below.)
2. The efficiency of the starter motor is 70 per cent.
3. The terminal voltage of the starter, which on open circuit is about 6 volts, drops to about 5 volts when the starter switch is closed as a result of internal losses in the battery caused by the large starting current.

The starter input power is

$$P_{\text{in.}} = \frac{420}{0.70} = 600 \quad \text{watts}$$

The starter current is

$$I_{\text{st.}} = \frac{600}{5} = 120 \quad \text{amp}$$

The power output of the starter is estimated on the following basis: In manual starting where a man exerts 50 lb force (assumed constant) at a 1-ft radius and spins the crank at the rate of 1 rps his power output is

$$\frac{2\pi \times 1 \times 50 \times 1.356}{1_{\text{sec}}} = 426 \quad \text{joules/sec or watts}$$

where the factor 1.356 converts the pound-feet (314 in this case) to joules. This rough estimate is rounded to 420 watts in the above example.

PROBLEMS

1. (*a*) What is the *potential energy* of mass m in Fig. 1, page 48, at the 1 position relative to the earth's surface if $m = 50$ kg and $h_1 = 10$ m?

(*b*) What is the gravitational *potential* of mass m if the earth's surface is reckoned as the zero potential plane?

Note: Since there is no accepted name for the mks unit of gravitational potential, express units in joules per kilogram.

2. An electrical generator, in transferring 2000 coulombs of positive charge from its negative terminal to its positive terminal (at a uniform time rate of transfer), requires a total energy input from its prime mover of 1 kwhr. If the generator is 90 per cent efficient, what is the potential of the positive terminal relative to the negative terminal during the period of time in which this transfer of charge takes place?

3. It will be assumed that the force experienced per positive coulomb of test charge in the A to B region shown in Fig. 7 may be expressed accurately enough for our purposes as

$$f = (80 - 20x) \quad \text{newtons} \qquad 0 < x < 2\text{ m}$$

where x is measured in meters, the origin of x being at the A position as indicated in Fig. 7. Note that this equation is qualified in such a manner that it is not to be used for x less than zero or for x greater than 2 m. What is the potential of point A relative to the potential of point B?

4. What is the absolute electrical potential (in volts) of a point which is 10^{-8} m from the center of a helium nucleus, assuming that the nucleus is a uniformly charged

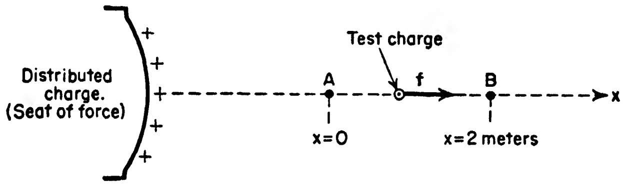


FIG. 7. See Prob. 3.

sphere of 10^{-14} m in diameter and that the nucleus is the only charge in the universe? A free space medium where $\epsilon_r = 1$ is assumed. *Ans.* 0.288 volt.

Note: This problem may be solved on an approximate basis similar to that shown on page 20 if, at this stage of his mathematical studies, the student does not appreciate the full significance of

$$W = \int_{10^{-8}}^{\infty} f\,dx$$

For all practical purposes, the *universe* in this particular case might be a sphere of about 0.1 mm (or even 0.01 mm) radius which encloses the nucleus since the force per positive coulomb of test charge turns out to be $f = (28.8 \times 10^{-10})/x^2$ newtons where x, the distance from the center of the nucleus, is expressed in meters.

5. What is the energy (or work) equivalent of 100 coulombs of positive charge falling through a potential drop of 135.6 volts expressed in pound-feet?

$$\frac{\text{(No. of) lb-ft}}{\text{joule}} = 0.7376$$

6. In a vacuum tube like that shown in Fig. 8, electrons are emitted at the cathode (as a result of heating the cathode by an electrical heater which is not shown). Once free from the cathode, the electrons are attracted to the positively charged plate, acquiring high velocities by the time they reach the plate. Upon striking the plate, the electrons give up their kinetic energy to the plate in the form of heat.

How many gram-calories of heat are developed at the plate of the tube as a result of the passage of 12.5×10^{18} electrons if the potential difference between the plate and the cathode is maintained at 200 volts as shown in Fig. 8?

$$\frac{\text{(No. of) g-cal}}{\text{joule}} = 0.2388$$

7. How many seconds are required to accomplish the work in Prob. 5 if the current is 10 amp?

8. (*a*) What is the magnitude of the *current passing through the tube* (in amperes) and the amount of *power dissipated at the plate of the tube* (in watts) in Prob. 6 if the 12.5×10^{18} electrons pass through the tube at a uniform time rate in 100 sec?

(*b*) What power is dissipated in element 1 of Fig. 8 throughout this period?

9. What is the direction of the *positive current equivalent* of the electron flow in the series loop shown in Fig. 8, clockwise or counterclockwise?

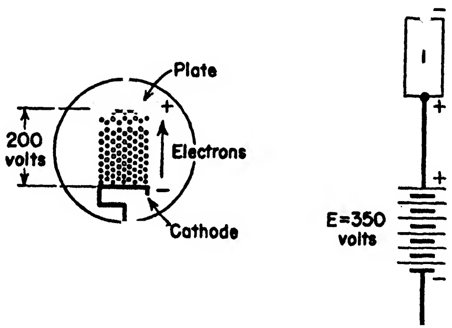


FIG. 8. See Probs. 6, 8, and 9.

10. The charge which passes the cross section of a conductor is known to oscillate back and forth across this cross section in such a manner that the charge *at* the cross section at any time can be expressed in equation form as

$$q = 0.2 \sin (377t) \quad \text{coulomb}$$

where time t is expressed in seconds, in which case $377t$ is in radians.

(*a*) What is the magnitude of the current at $t = 0$; at $t = \frac{1}{240}$ sec; and at $t = \frac{1}{120}$ sec?

(*b*) Why is it that maximum current occurs when $q = 0$, and that zero current occurs when q is at its maximum value?

11. The current in a particular circuit is known to follow a sinusoidal time variation which can be written in equation form as

$$i = I_m \sin \omega t \quad \text{amp}$$

and the voltage drop across a dissipative element in this circuit is known to be

$$v = V_m \sin \omega t \quad \text{volts}$$

where I_m and V_m are the maximum values of the sinusoidal variations, and ω is the angular frequency of the variations expressed in radians per second. (In expressions like $I_m \sin \omega t$, t is expressed in seconds, in which case ωt becomes angular measure in radians.)

Find the power dissipated in the circuit element at $t = 0.0125$ sec and at $t = 0.025$ sec, if $I_m = 10$ amp, $V_m = 150$ volts, and $\omega = 62.8$ radians/sec.

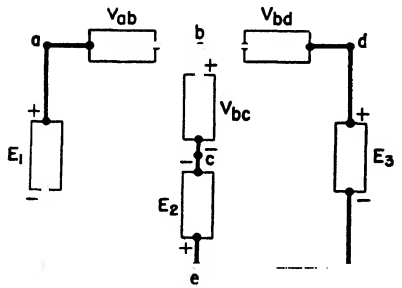


FIG. 9. See Prob. 12.

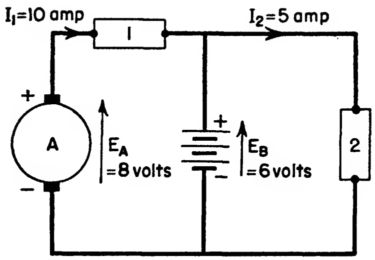


FIG. 10. See Prob. 13.

12. In Fig. 9, the three emf's are $E_1 = 100$, $E_2 = 200$, and $E_3 = 400$ volts. The elements ab, bc, and bd are dissipative circuit elements. If $V_{bc} = 150$ volts, find the magnitude and polarity of V_{ab}; of V_{db}.

13. (*a*) What is the power delivered by the generator E_A of Fig. 10?

(*b*) Is the battery E_B delivering or absorbing power and to what extent?

(*c*) What power is absorbed by the dissipative elements 1 and 2?

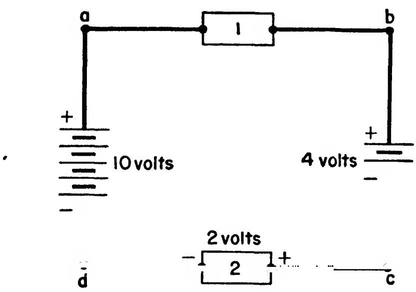


FIG. 11. See Prob. 14.

14. What is the potential difference between points a and b of Fig. 11 and between points d and b? Determine the latter by way of the dab path and by way of the dcb path.

15. (*a*) In Fig. 12, what are the potentials (or voltages) of the following points relative to ground potential: point g, point k, and point p?

(*b*) What is the potential *rise* from k to g?

(*c*) What is the potential difference between points p and k, and which is at the higher potential?

16. If in Fig. 13, $\text{emf}_1 = 110$ volts, $\text{emf}_2 = 130$ volts, $V_{cb} = 30$ volts, $I_{dab} = 10$ amp, and $I_{dcb} = 5$ amp; find $P_{\text{in}1}$, $P_{\text{in}2}$, $P_{\text{out}1}$, $P_{\text{out}2}$, and $P_{\text{out}3}$.

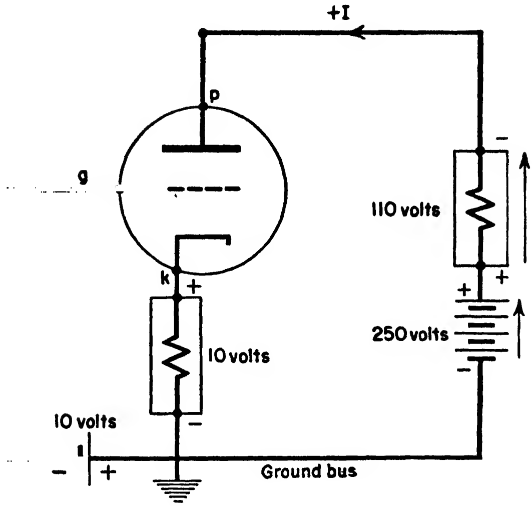


FIG. 12. See Prob. 15.

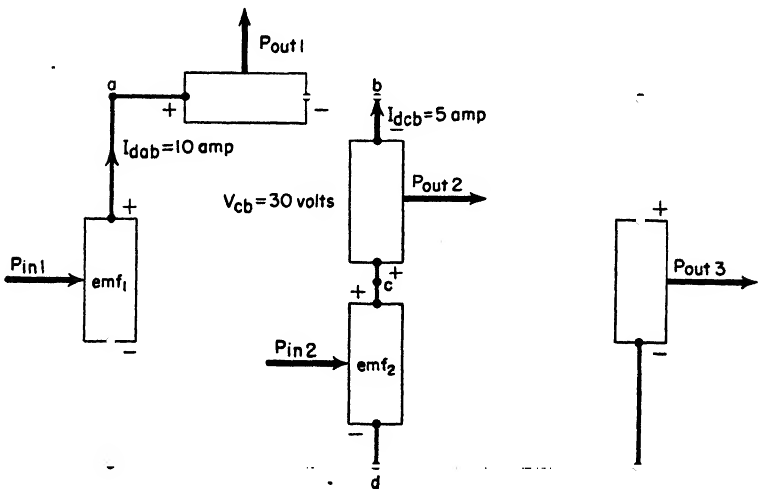


FIG. 13. See Prob. 16.

17. In Fig. 14, the single emf shown develops a voltage rise of 10 volts, namely, $E_{da} = 10$ volts. $I_{ab} = 1$ amp, $I_{ac} = 2$ amp and $V_{ab} = V_{ac} = 4$ volts. A, B, C, D, and E are dissipative elements.

Find P_{in}, P_A, P_B, P_C, P_D, and P_E in watts.

18. In Fig. 15-a are shown three switches (1, 2, and 3), all in their *a* positions, and in Fig. 15-b are shown the same three switches in their *b* positions. [The lamp is plainly energized with all the switches in the *a* position (as in Fig. 15-a) and de-energized with all the switches in the *b* position (as in Fig. 15-b)].

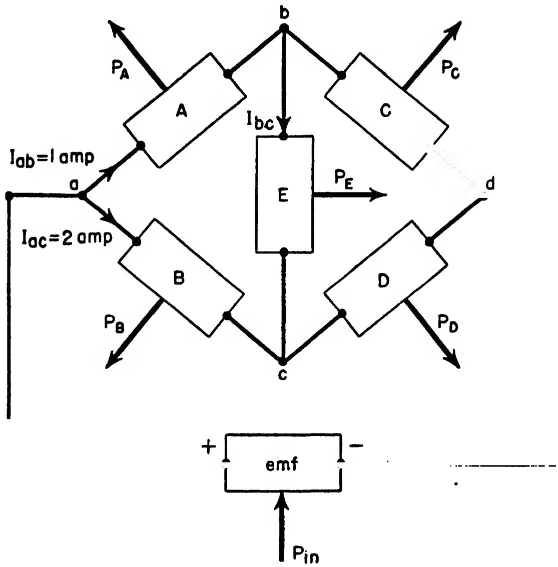


FIG. 14. See Prob. 17.

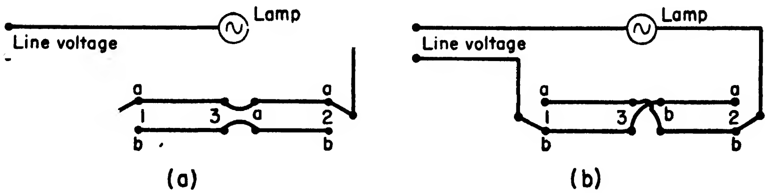


FIG. 15. See Prob. 18.

(*a*) Show that the lamp is de-energized by switching any one of the three switches from the *a* position (in Fig. 15-a) to the *b* position.

(*b*) Show that the lamp is energized by switching any one of the three switches from the *b* position (in Fig. 15-b) to the *a* position.

(*c*) Draw a circuit diagram whereby a lamp can be either energized or de-energized from any one of four different places or switching positions.

CHAPTER IV

Electrical Resistance and Resistivity

1. Preliminary Considerations. If a potential difference V is applied to the end faces of a piece of material as shown in Fig. 1, the magnitude of the current that will flow through the material will depend upon several factors:

(1) The length of the conductor, l.
(2) The cross-sectional area of the conductor, A.
(3) The molecular structure of the material.
(4) The temperature of the material.
(5) The potential difference between the end faces, V.

If the material is metal, for example, the current will be roughly 10^{19} times greater than if the material is glass (operating at room temper-

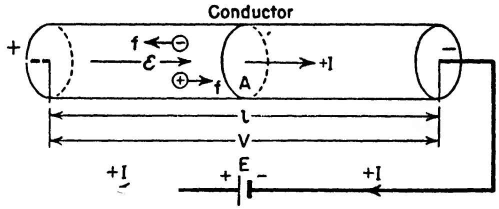


FIG. 1. Illustrating $\mathcal{E} = \left.\frac{dV}{dl}\right]_{\max.} = \frac{f}{Q}$.

ature), if the other factors are about the same in the two cases. The nuclei of metallic atoms hold their outermost orbital electrons so loosely that these electrons drift toward the positive end of the rod in Fig. 1. The negative charge which moves from $-$ to $+$ through the metal is precisely equivalent to a corresponding positive charge moving in the $+$ to $-$ direction.

The equivalent $+I$ as shown in Fig. 1 is usually employed in circuit analysis even though the actual transportation of charge in metallic conductors is the result of electron (or negative charge) movement. In some conductors, positive and negative charges move in opposite directions to produce a resultant current which can be accounted for in terms

of positive charge movements (dq_1^+/dt) and negative charge movements (dq_2^-/dt) as follows:

$$\text{resultant } I = \frac{dq_1^+}{dt} \text{ (from } + \text{ to } -) + \frac{dq_2^-}{dt} \text{ (from } - \text{ to } +) \quad (1)$$

The dq/dt's written above refer to time rates of change of electric charge across a specified cross-sectional area like A in Fig. 1.

Even though the two components of the resultant current in equation (1) actually exist in some types of conductors, the negative charge component dq_2^-/dt (from $-$ to $+$) is usually replaced by its equivalent dq_2^+/dt (from $+$ to $-$) to obtain an equivalent positive current. In equation form,

$$\begin{aligned} \text{equivalent } +I &= \left(\frac{dq_1^+}{dt} + \frac{dq_2^+}{dt}\right) \text{ (from } + \text{ to } -) \\ &= \frac{dq_{\text{eq.}}^+}{dt} \text{ (from } + \text{ to } -) \end{aligned} \quad (2)$$

Since the use of the equivalent $+I$ simplifies the concept of current flow, it is widely used in circuit analysis.

2. The Electric Intensity Vector $\mathcal{E}$. Conduction phenomena can usually be stated more clearly and concisely in terms of the space rate of change of potential drop ($dV/d\mathbf{l}$) than in terms of potential drop V itself. In Chapter III it was shown that

$$V = \frac{\text{energy}}{\text{charge}} = \frac{W}{Q} \quad (3)$$

The differential potential drop between two points $d\mathbf{l}$ distance apart is therefore

$$dV = \frac{dW}{Q} = \frac{\mathbf{f} \cdot d\mathbf{l}}{Q} \quad (3\text{–a})$$

It follows that

$$\left.\frac{dV}{d\mathbf{l}}\right]_{\text{max.}} = \frac{\mathbf{f}}{Q} = \mathcal{E} \quad (4)$$

where $\mathbf{f}$ is the force acting on charge Q and $\mathbf{l}$ is the displacement through which the force acts. The subscript "max." is simply a reminder that the direction of the displacement, $\mathbf{l}$, must be so chosen that the maximum space rate of change of potential drop is encountered.

Equation (4) defines the electric vector $\mathcal{E}$ at any point both as the maximum space rate of change of potential drop at that point and as the electric force per unit charge at that point. For the present we shall

be chiefly interested in the latter definition since it has a special significance as applied to the movement of electric charge. (In Chapter VII, the electric vector, $\mathcal{E}$, sometimes called the *electric intensity vector*, will be considered in more detail.)

In applying equation (4) to Fig. 1, we note that l is measured along a straight line running from the + end of the conductor to the − end. If, for example, the rod is 5 m long and the potential drop from the + end to the − end is 0.20 volt, the maximum space rate of change of potential drop along the conductor is

$$\mathcal{E} = \left.\frac{dV}{dl}\right]_{\text{max.}} = \frac{0.20}{5} = 0.04 \quad \text{volt/m}$$

since a uniform medium is assumed to exist between the two end faces of the conductor. In this case the electric vector $\mathcal{E}$ is directed along the axial length of the conductor as shown in Fig. 1.

If a charge carrier is present in a region where $\mathcal{E} = \left.\frac{dV}{dl}\right]_{\text{max.}}$ exists, the charge carrier will have an electrical force developed upon it in accordance with equation (4). If the carrier is positively charged, the electrical force tends to move the carrier in the $+\mathcal{E}$ direction; if the carrier is negatively charged, the electrical force tends to move the carrier in the $-\mathcal{E}$ direction. (See Fig. 1.) Whether or not the charge carriers actually migrate along the axial length of the conductor depends upon the atomic structure of the material.

Example. Consider an electron in the region between two flat parallel plates which are separated from one another by a distance of 2 cm and which are maintained at a potential difference of 200 volts.

The electron ($Q_e = -1.6 \times 10^{-19}$ coulomb) will have a force exerted on it of

$$\mathbf{f} = \mathcal{E}Q_e = \frac{200}{0.02} \times 1.6 \times 10^{-19} = 1.6 \times 10^{-15} \quad \text{newton}$$

This force is directed from − to +, that is, in the $-\mathcal{E}$ direction.

If the electron is wholly unrestrained, it will experience an acceleration of

$$a = \frac{f}{m_e} = \frac{1.6 \times 10^{-15}}{9.1 \times 10^{-31}} = 1.76 \times 10^{15} \quad \text{m/sec}^2$$

Whereas the maximum space rate of change of *potential drop*, $\left.\frac{dV}{dl}\right]_{\text{max.}}$, is by definition the electric vector $\mathcal{E}$, the *potential gradient* is by definition the maximum space rate of change of *potential rise*, $\left.\frac{dE}{dl}\right]_{\text{max.}}$. Except

where the direction of voltage *drop* is to be distinguished from the direction of voltage *rise*, potential gradient signifies the same thing as does the electric vector $\mathcal{E}$. That is,

$$\text{potential gradient} = \left.\frac{dE}{dl}\right]_{\text{max.}} = -\mathcal{E} \tag{4-a}$$

3. Insulators. Some materials, under normal working conditions, permit an inappreciable amount of electric charge to pass through them. These materials are called insulating materials. Some of the more common insulators are

Glass	Rubber	Paper
Mica	Ebonite	Bakelite
Polystyrene	Cloth	Asbestos

Air, under normal conditions, is a good insulator, as is distilled water. Certain grades of oil also have excellent insulating properties.

With the aid of insulators, electric currents can be confined almost exclusively to the specified paths which we call circuits. The molecules (or groups of molecules) in insulating materials hold the orbital electrons so rigidly that movement of charge within these materials can be effected only with very high potential gradients. Any material, however, can be broken down or punctured if subjected to sufficiently high potential gradients. The breakdown (or dielectric) strengths of a few materials are shown below. Since the actual value of potential gradient at which a material breaks down depends to a large measure upon the test conditions actually employed, the figures given should be accepted as approximate representative values applying to relatively thin specimens of the material.

Material	Dielectric Strength (volts per millimeter)
Air (atm. pr., 0°C)	3,000
Asbestos paper	4,200
Wood (maple)	4,600
Paper	8,700
Oil (insulating)	15,000
Mica	25,000
Bakelite (Micarta-213)	31,000
Cloth (empire)	48,000
Rubber (hard)	70,000

If the surface of an otherwise good insulator becomes dirty, moist, or oxided a great deal more current is likely to flow over the surface than through the body of the material. Some materials which, under normal

operating conditions, are good insulators may, under abnormal conditions, become fairly good conductors. If, for example, air is broken down by a potential gradient which exceeds its dielectric strength an arc is formed which is a relatively good conductor. Certain grades of glass rod which under normal conditions are good insulators will, if heated to a dull red, become fairly good conductors.

No sharp distinction can be made between insulators and conductors because in certain applications a specified element may be a better conductor than is required whereas the same element in another application may be a better insulator than is required.

4. Conduction Processes. The electrical engineer is primarily concerned with the transportation of electric charge through

(1) Ionized solutions (like salt and acid solutions).

(2) Vacuum tubes (glass or metal envelopes evacuated except for charge carriers).

(3) Carbon, mercury, and solid metals.

Electrolytic Conduction. A water solution of NaCl (common salt) will, if subjected to a potential difference as shown in Fig. 2, conduct electric charge because charge carriers which are free to move are present in the solution. A molecule of NaCl in the presence of water will separate into ionized atoms, Na^+ and Cl^-. The symbol Na^+ indicates that the sodium atom is positively charged, and the symbol Cl^- indicates that the chlorine atom is negatively charged. As a result of the separation of NaCl in the presence of water, the sodium atom loses its one M-energy-level electron which the chlorine atom acquires. (See Appendix C.) The sodium atom is left with a net positive charge of 1.6×10^{-19} coulomb, and the chlorine atom has a net negative charge of 1.6×10^{-19} coulomb. The $+$ ions (Na^+) move in the $+\mathcal{E}$ direction, and the $-$ ions move in the $-\mathcal{E}$ direction. These movements of charge result in an equivalent $+I$ which flows counterclockwise around the circuit shown in Fig. 2. See equations (1) and (2).

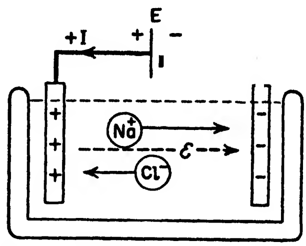


FIG. 2. Electrolytic conduction.

Since the charge carriers in the solution are atomic in size, they cannot pass through the metal electrodes or metallic wires which connect the battery (E) to the ionized solution. Each positive ion (Na^+) upon arriving at the negative electrode extracts an electron from the metallic circuit and in so doing becomes a normal or un-ionized sodium atom. Each negative ion (Cl^-) upon arriving at the positive electrode gives

up one electron and becomes a normal chlorine atom. Thus particles, atomic in size, transport the charge through the electrolyte, and the current throughout the series circuit is rendered continuous.

Electron Current in a Vacuum Tube. If two electrodes are placed in an evacuated envelope as shown in Fig. 3, and the cathode heated to a sufficiently high temperature, electrons will be emitted from the cathode. These electrons are attracted by the positive plate and travel from the negative cathode to the positive plate, thus constituting a flow of current through the tube.

If the envelope were perfectly evacuated, no positive charge carriers would be present, and the entire conduction process between the

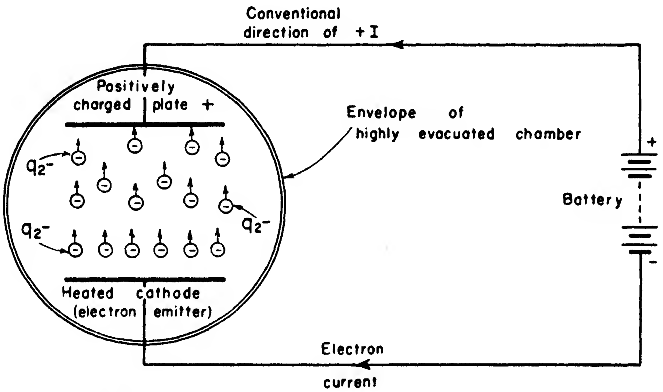


FIG. 3. Illustrating conduction in a high-vacuum tube with electrons as charge carriers.

electrodes would be accounted for in terms of the electrons which move from cathode to plate. This type of conduction is essentially that which takes place in a high-vacuum tube although some few positive charge carriers are present because perfect evacuation cannot be obtained in practice. Even though the entire conduction process in a vacuum tube is accomplished by electron movement, it is customary to account for this movement in terms of an equivalent $+I$ as shown in Fig. 3.

In Fig. 2, current will flow through the ionized solution regardless of which electrode is connected to the positive terminal of the battery E. In Fig. 3, a current will flow only if the plate is made positive relative to the heated electrode. The vacuum tube is essentially a unilateral conductor of electricity because, if the cold electrode (plate) were negative relative to the heated electrode, the emitted electrons would tend to return to the heated electrode.

The current which flows in a vacuum tube (connected as shown in

Fig. 3) is not linearly related to the potential difference between the electrodes. Because the heated electrode can supply only a limited number of emitted electrons, a saturation effect occurs as indicated in Fig. 4. As the potential difference V is increased from zero value, the current varies roughly as the 1.5 power of V until saturation effects come into play. For higher values of V, the increase in current is relatively small. The tube is therefore a circuit element which requires careful consideration in circuit analysis.

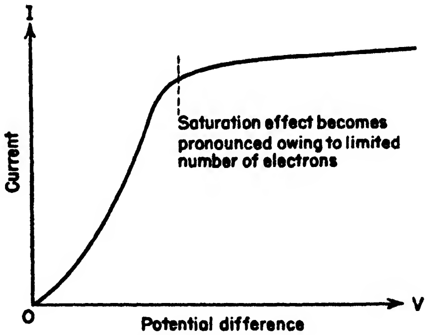


FIG. 4. Illustrating the saturation effect in the current of a high-vacuum tube.

Metallic Conduction. In a metal, the atoms are usually grouped in orderly arrangements called crystals. A cross-sectional view of a crystalline structure is indicated schematically in Fig. 5. If the nuclei are those of sodium, for example, each nucleus will be surrounded by two K-energy-level electrons, eight L-energy-level electrons, and *one* M-energy-level electron. (See Appendix C.) Owing to the symmetrical arrangement of the nuclei, an M-energy-level electron can find itself acted upon by four nuclei simultaneously as indicated in Fig. 5. The result is that this electron is essentially "free" as compared with electrons in the L or K levels. Free electrons in a metal wander from one atom to another and, because of their thermal energy, produce random currents within the interior of the metal even though no potential difference is applied to the conductor. This random movement of charge results in no useful current because as many electrons in any small region of the conductor move in one direction as in the opposite direction during an increment of time.[1]

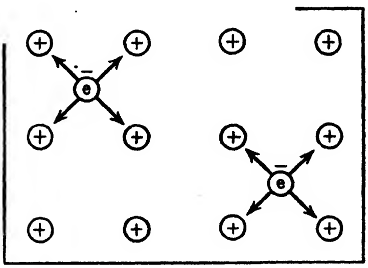


FIG. 5. Schematic representation of free electrons (e) in a crystalline structure.

[1] At any instant, however, a net movement of electrons may exist in one direction, and at the next instant a net movement may exist in the opposite direction giving rise to an alternating current within the conductor. Since this type of movement is purely haphazard in its alternations, the resulting current is known as *noise*

Reference to Appendix C will show that metals like sodium, potassium, and rubidium have only one electron in the outermost energy level. These metals (including lithium and cesium) are highly electropositive; in other words, they hold the outermost energy-level electron very loosely. Cesium is the best conductor known; rubidium, potassium, sodium, and lithium follow. These metals are not ordinarily used as conductors because they are highly reactive chemically. They oxidize and disintegrate in air and react violently with water. Other metals like copper, aluminum, tungsten, nickel, and silver are relatively stable and are good conductors of electricity.

Theoretical considerations, supported by some experimental evidence, leads us to believe that every atom (or possibly every three or four atoms) of a good conductor like copper or aluminum provides one *free* electron. In any conductor of ordinary dimensions, there is an enormously large number of free electrons. A cubic centimeter of copper, for example, contains about 8.4×10^{22} atoms of copper and a correspondingly large number of free electrons.

If the conductor shown in Fig. 1 is a metallic conductor, the free electrons move along the conductor in the $-\mathcal{E}$ direction. The atomic nuclei do not enter into the conduction process, and no chemical effects result from the movement of the free electrons. No saturation effects of the kind shown in Fig. 4 are observed in metallic conduction because of the copious supply of free electrons.

5. Current Density. In dealing with conduction problems, it is often more convenient and informative to deal with current per unit cross-sectional area than to deal with the total current. Current per unit area is called *current density*, and it is usually symbolized by J. As applied to any cross-sectional area like that shown in Fig. 1,

$$J = \frac{I}{A} \tag{5}$$

The relationship between the current density and the volume density of charge carriers may be derived quite simply by assuming that the total charge Q, in a conductor of Δl length, passes the cross-sectional area A in time Δt. Thus

$$\mathbf{J} = \frac{I}{\mathbf{A}} = \frac{Q}{\mathbf{A}\,\Delta t} = \frac{(Nq) \times \mathbf{A} \times \Delta \mathrm{l}}{\mathbf{A}\,\Delta t} = Nq\mathbf{v} \tag{6}$$

current or simply *noise* because these effects are of paramount importance in amplifying very weak signals like those generated by microphones and similar sources. Noise currents impose a lower limit on the strength of a signal which can be amplified because, if the desired signal is not sufficiently greater than the noise signal, the output of the amplifier will be unintelligible.

where N is the number of charge carriers per unit volume

q is the charge of each charge carrier

$\mathbf{v}$ is the velocity with which these charge carriers pass the cross-sectional area **A**.

If both positive and negative charge carriers enter into the conduction process, equation (6) may be written as

$$\mathbf{J} = N_1 q_1^+ \mathbf{v}_1^+ + N_2 q_2^- \mathbf{v}_2^- \tag{7}$$

where the 1 subscripts refer to positive charges and the 2 subscripts to negative charges. In general $\mathbf{v}_2^-$ is directly opposite to $\mathbf{v}_1^+$, so the resultant current density is simply the arithmetic sum of the two components, $N_1 q_1^+ \mathbf{v}_1^+$ and $N_2 q_2^- \mathbf{v}_2^-$. In effect $N_2 q_2^- \mathbf{v}_2^-$ is equivalent to $N_2 q_2^+ \mathbf{v}_2^+$ because q_2^- traveling in the $\mathbf{v}_2^-$ direction is equivalent to a corresponding positive charge q_2^+ traveling in the opposite direction, namely, the $\mathbf{v}_2^+$ direction.

Where both types of charge carriers are present like in electrolytes and gas tubes, the positive and negative charges usually travel at different velocities ($v_1 \neq v_2$) because of their different masses. The charge carriers usually possess charge in the amount of one, two, or three electronic charges; therefore in a region of specified potential gradient both the positive and negative charge carriers have forces developed on them which are of the same order of magnitude. Since the positive charge carriers met with in practice are of atomic weight, they will be thousands of times more massive than an electron itself, with the result that the positive charge carriers will be accelerated much less than the electrons.

Example. Let it be required to calculate the velocity of the electron drift [v_2^- in equation (7)] in copper if the current density is 1000 amp/sq in., assuming that each four atoms of copper contribute one free electron to the conduction process. Since $\mathbf{v}_1^+ = 0$, the specified current density is due entirely to electron drift.

$$J = \frac{1000}{6.45} \text{ amp/sq cm} \qquad N_2 = \frac{8.4 \times 10^{22}}{4} \text{ electrons/cu cm}$$

$$q_2^- = -1.6 \times 10^{-19} \text{ coulomb} \quad N_2 q_2^- = -3.36 \times 10^3 \text{ coulombs/cu cm}$$

$$\mathbf{v}_{2\text{avg.}}^- = \frac{1000}{6.45} \times \frac{1}{-3360} = -0.0462 \text{ cm/sec}$$

where the minus sign indicates that the electrons are traveling in the $-\mathcal{E}$ direction, that is, directly opposite to the positive current density direction which is in the $+\mathcal{E}$ direction.

6. Ohm's Law. In metallic conductors, a straight-line relationship exists between the current density (J) and (dV/dl) as indicated in Fig. 6. For a conductor of uniform cross-sectional area, the experimentally determined relationship shown in Fig. 6 may be expressed as

$$J = \gamma \frac{V}{l} = \frac{1}{\rho} \frac{V}{l} \tag{8}$$

where γ is the conductivity of the conductor material
$\rho = 1/\gamma$ is the resistivity of the conductor material
V is the voltage drop across l units of length of conductor.

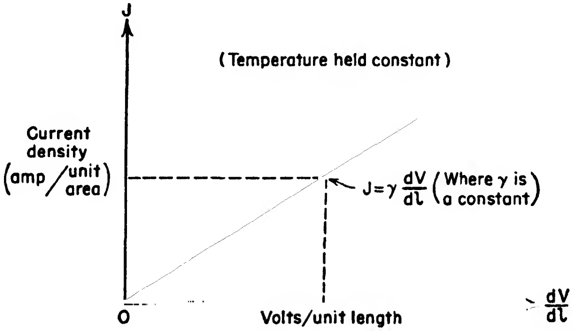


FIG. 6. Linear relationship of J and dV/dl.

If both sides of equation (8) are multiplied by the cross-sectional area of the conductor, A, the equation takes the form

$$I = \frac{V}{\rho \dfrac{\text{length (of conductor)}}{\text{area (of conductor)}}} = \frac{V}{R} \tag{9}$$

where $R = \rho(l/A)$ is called the resistance of the conductor. The unit of resistance normally used in practice is the ohm because

$$\frac{\text{volts}}{\text{amp}} = \text{ohms} \tag{9-a}$$

Equation (9) is usually referred to as Ohm's law because Ohm (about 1826) discovered this relationship experimentally. This law is very useful in elementary circuit analysis, but the student is cautioned against using it indiscriminately. Only in resistive circuit elements where I and V are related linearly can equation (9) be used directly.

7. Resistance and Resistivity. The current I in equation (9) refers to the total current across the cross-sectional area A of the conductor. For the present, A is assumed to be the same over the entire length of the conductor, that is, A is not a function of l. The voltage drop V in equation (9) applies to the total length of conductor l. The dimensions of the conductor are all contained in the resistance factor R of equation (9). That is,

$$R = \rho \frac{l}{A} \tag{10}$$

Resistance is a circuit parameter which opposes the flow of current and causes a non-reversible transformation of electrical energy to heat energy within the conductor.

The resistivity ρ is dimensionally

$$\frac{\text{resistance} \times \text{area}}{\text{length}}$$

as may be seen from equation (10), and it depends for its value upon the conductor material and the units employed in expressing resistance, area, and length. In practice, ρ is usually expressed in one of the three following ways:

(1) Ohm-centimeters (or ohms per centimeter cube.)
(2) Ohm-inches (or ohms per inch cube.)
(3) Ohms per circular-mil-foot (or ohm-circular-mils per foot).

By the ohm-centimeter is meant the resistance in ohms of a specimen which is 1 sq cm in cross-sectional area and 1 cm in length. This specimen is geometrically a cube and, although the term "ohms per centimeter cube" is dimensionally incorrect, it appears frequently in the literature and is generally acceptable.

If the length of the specimen is 1 ft and the cross-sectional area is 1 cir mil, ρ is expressed correctly in ohm-circular-mils per foot, although common usage has made the term "ohms per circular-mil-foot" acceptable. A circular mil is the area of a circle one mil (0.001 in.) in diameter. It is a convenient unit of area to employ in dealing with wires of circular cross section because the area of round wires is simply $(D_{\text{mils}})^2$ cir mils. Wire tables usually tabulate the area of circular wire in circular mils. (See Appendix A.) A graphical explanation of a circular mil is shown in Fig. 7.

Exercise. Show that the *number* of circular mils per square centimeter is 1.973×10^5.

Example. As shown in Table I, commercial annealed copper has a resistivity of 1.73×10^{-6} ohm-sq cm/cm or 1.73×10^{-6} ohm/cm cube. The resistivity of this copper expressed in ohm-circular-mils per foot (or in ohms per circular-mil-foot) is

$$\rho = 1.73 \times 10^{-6} \times 30.48 \times 1.973 \times 10^{5} = 10.4 \text{ ohms/cir-mil-ft at } 20°\text{C}$$

since the circular-mil-foot specimen is 30.48 times longer and 1.973×10^{5} times less in cross-sectional area than the centimeter cube specimen.

The figure $30.48 \times 1.973 \times 10^{5}$ ($=6.014 \times 10^{6}$) converts resistivity in ohms per centimeter cube to ohms per circular-mil-foot and is useful where this type of conversion has to be made.

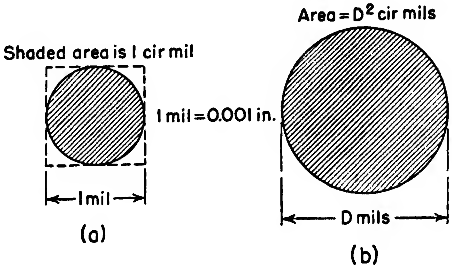


FIG. 7. Illustrating area $_{\text{cir mils}} = (D_{\text{mils}})^2$.

Exercise. Express the resistivity of the copper described above in microhms per inch cube. (A microhm is one millionth of an ohm.)

Ans. 0.681 microhms/in. cube.

8. Resistivities of Various Metals. In Table I are listed the resistivities of several metals and metal alloys. In general, metal alloys have resistivities which are considerably greater than the resistivities of the pure metals out of which the alloys are formed. Materials having high resistivities when made up into the form of wire are called resistance wire. Resistance wire is obtained under trade names like Nichrome, which is a nickel-chromium alloy, Advance and Constantan, which are copper-nickel alloys, and so on.

Silver has the lowest resistivity of the metals listed in Table I. The cost of silver, however, prohibits its use as an electrical conductor except in special cases. In ultra-high-frequency wave guides where the current is confined largely to a very thin surface layer, silver-plated conductors are widely used. In the manufacture of the mass spectographs used in connection with the atomic bomb project, tons of silver were used in the coils of the electromagnets because the Government had plenty of silver in its monetary reserve, whereas copper was a critically short material during the war.

TABLE I

Material	Resistivity at 20° C		Resistivity Temperature Coefficient/ Degree Centigrade at 20°C
	Microhm-cm or microhm/ cm cube	Ohms/ cir-mil-ft	
Silver	1.63	9.8	+0.0038
Copper (pure annealed)	1.69	10.2	
Copper (commercial annealed)	1.73	10.4	0.00393
Copper (hard-drawn)	1.77(6)	10.6	
Aluminum (commercial)	2.83	17.0	0.0039
Tungsten (annealed)	4.37	26.3	0.0045
Zinc	6	36	0.0037
Iron (99.97% pure)	9.8	59	0.006
Platinum	10	60	0.003
Nickel (commercial)	11	66	0.004
Lead	22	132	0.0039
Mercury	95.78	576	0.00089
German silver (Ni 18, Cu 65, Zn 17%)	29.1	175	0.00027*
Manganin (Ni 4, Cu 84, Mn 12%)	48.3	290	±0.00001*
Nickel copper (Advance) (Ni 45, Cu 55%)	48.9	294	±0.00001*
Nickel chromium (Ni 80, Cr 20%)	108	650	0.0001*

* Actual values vary considerably with small variations in the composition. All values shown in the table vary somewhat with the manufacturing techniques employed.

Copper and aluminum are the materials most widely used for electrical conductors where low resistivity is desired. Even though aluminum has a resistivity which is about 65 per cent greater than that of copper, the lighter weight of aluminum is sometimes in its favor. Other things being equal, aluminum is as cheap a conductor as copper if its price per pound is not more than about twice the price of copper. Aluminum is much more difficult to solder than is copper. Where space is limited, as in motor and generator windings, copper is used almost exclusively.

Where tensile strength is of importance, as in overhead transmission lines, hard-drawn copper is often used. The resistivity of hard-drawn copper is about 2 or 3 per cent greater than that of annealed copper but its tensile strength is more than double that of annealed copper.

Example. Let it be required to find the resistance of a magnet coil which is wound with 250 ft of No. 20 B. & S. gage copper wire. It is known that the resistivity of the copper is 10.4 ohms/cir-mil-ft at 20°C. Reference to Appendix

A will show that the diameter of No. 20 gage wire is 31.96 mils (0.03196 in.) in diameter and that the cross-sectional area is 31.96^2 or 1022 cir mils.

$$R = \rho \frac{l}{A} = 10.4 \frac{250}{1022} = 2.54 \quad \text{ohms (at 20°C)}$$

9. Change of Resistance with Change of Temperature. The temperature coefficient of resistivity at any temperature, say T_1, is defined as

$$\alpha_1 = \frac{1}{\rho_1} \frac{d\rho}{dT}\bigg]_{\text{evaluated at } T_1} \tag{11}$$

where ρ_1 is the resistivity at temperature T_1

T_1 is the temperature (usually expressed in centigrade degrees).

Equation (11) may be written in terms of finite differences as

$$\alpha_1 = \frac{1}{\rho_1} \frac{\Delta\rho}{\Delta T} = \frac{1}{\rho_1} \frac{\rho_2 - \rho_1}{T_2 - T_1} \tag{12}$$

where ρ_2 is the resistivity at temperature T_2.

The temperature coefficient of resistivity is the per unit change in resistivity per degree change in temperature.

Over the range of temperatures normally employed in practice (−30°C to 100°C) the resistivity of metals varies approximately linearly with temperature. This means that $d\rho/dT$ in equation (11) is essentially constant, but the value of α will depend upon the value of ρ employed in specifying α. In other words, α is a function of temperature since ρ is a function of temperature.

Equation (12) may be rearranged as

$$\rho_2 = \rho_1[1 + \alpha_1(T_2 - T_1)] \tag{13}$$

The above form is useful in finding ρ_2 if α_1 is known at the starting temperature T_1. A more convenient relationship is established by noting from equation (13) that

$$\rho_T = \rho_0(1 + \alpha_0 T) \tag{14}$$

where T is now any temperature reckoned from 0°C. Employing equation (14) for two different temperatures T_2 and T_1, it is plain that

$$\frac{\rho_2}{\rho_1} = \frac{1 + \alpha_0 T_2}{1 + \alpha_0 T_1} = \frac{\dfrac{1}{\alpha_0} + T_2}{\dfrac{1}{\alpha_0} + T_1} \tag{15}$$

Since the zero-degree temperature coefficient of copper is about 0.00427, equation (15) becomes

$$\frac{\rho_2}{\rho_1} = \frac{234.5 + T_2}{234.5 + T_1} \quad \text{(for copper at ordinary temperatures)} \qquad (16)$$

If any particular copper conductor is under consideration, the length and cross-sectional area is the same; therefore the ρ's in equation (16) may be replaced with resistances as

$$\frac{R_2}{R_1} = \frac{234.5 + T_2}{234.5 + T_1} \quad \text{(for copper at ordinary temperatures)} \qquad (17)$$

where R_2 is the resistance at temperature T_2°C
R_1 is the resistance at temperature T_1°C.

Several pure metals have positive temperature coefficients of resistivity which are roughly equal to that of copper. (See Table I.) A positive coefficient means that the resistivity (and hence resistance) increases with an increase of temperature. Carbon and ceramic materials (as well as electrolytes) possess negative temperature coefficients of resistivity within ordinary temperature ranges, which means that the resistance decreases with an increase of temperature. The negative temperature coefficient of ceramic resistors is sometimes used to compensate for the positive temperature coefficient of metallic resistors in bridges and other types of electrical apparatus. Negative temperature coefficient resistors (sometimes called *thermistors*) are considered in some detail in Chapter XV.

Reference to Table I will show that alloys like Manganin and nickel-copper have very low temperature coefficients of resistivity. Some alloys actually have small negative temperature coefficients within certain small ranges of temperature.

Example. Let it be required to find the resistance of the magnet coil in the example on page 82 at 75°C. The resistance at 20°C was found to be 2.54 ohms. From equation (17), it is plain that

$$R_{75^\circ} = \frac{234.5 + 75}{234.5 + 20} \times 2.54 = 1.216 \times 2.54 = 3.09 \quad \text{ohms}$$

It will be seen that a 21.6 per cent increase in resistance results from the 55°C increase in temperature from 20°C to 75°C. It turns out that the ratio of the 100°C resistance of several pure metals (like silver, copper, aluminum, zinc, and lead) to the 0°C resistance is approximately $\sqrt{2}$.

In specifying the resistance of a copper (or other pure metal) conductor, the working temperature must be specified if other than a rough approximation is intended.

Equation (17) is sometimes used to determine the working temperature of copper conductors in electrical apparatus. R_1 is measured at room temperature T_1 before the apparatus is operated. Immediately after a period of operation, R_2 is measured and T_2 computed with the aid of equation (17).

10. Resistance where Area A Varies with Length l. The expression $R = \rho l/A$ applies to an entire conductor only where the cross-sectional

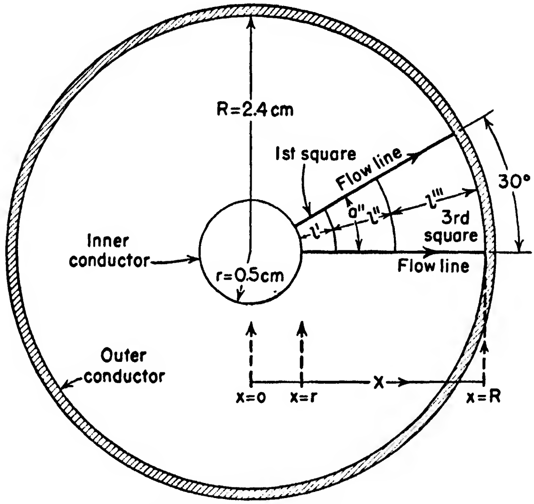


FIG. 8. Determination of insulation resistance.

area (which faces the flow of current) is uniform, that is, where A is not a function of l.

As a simple example of a case where A is a function of l, consider the insulation resistance between the inner conductor and the outer conductor of the coaxial cable shown in Fig. 8. If the inner conductor and the outer conductor are maintained at a potential difference, a small leakage current will flow through the insulation material along the *flow* lines as indicated. As applied to the insulation resistance, l is directed along radial lines between the conductors, and it will be observed that, between the two 30°-displaced current (or flow) lines, the area facing the current flow increases as the distance from the inner conductor increases.

A general expression for the resistance of an infinitesimal length, dl,

between the two 30°-displaced current lines shown in Fig. 8 is

$$dR_{30°} = \rho \frac{dl}{a} = \rho \frac{dx}{\frac{2\pi x}{12} L} \qquad R > x > r \tag{18}$$

where ρ is the resistivity of the insulation material, assumed constant
a is the area at any distance x from the center of the inner wire
x is measured radially between the conductors as shown in Fig. 8
L is the length of the cable, measured at right angles to the plane of the cross section shown in Fig. 8.

The factor 12 which appears in equation (18) accounts for the fact that only the resistance between the two 30°-displaced current lines is being calculated at this time. It will be noted that there are 12 current paths (like the one shown in Fig. 8) in parallel between the inner and outer conductors. The reason for making this arbitrary breakup will be evident presently.

Returning now to equation (18), the actual resistance is obtained by summing up all the infinitesimal dR's between $x = r$ and $x = R$. This may be done formally by calculus as

$$R_{30°} = \frac{12\rho}{2\pi L} \int_r^R \frac{dx}{x} = \frac{12\rho}{2\pi L} \ln \frac{R}{r} \tag{19}$$[2]

or it may be done by a graphical method which involves no calculus. The actual insulation resistance between the conductors of Fig. 8 is $\frac{1}{12}$ that given in equation (19) because there are 12 $R_{30°}$'s in parallel between the conductors. A general expression for the insulation resistance of a coaxial cable of the kind shown in Fig. 8 is

$$R_{\text{ins.}} = \frac{\rho}{2\pi L} \ln \frac{R}{r} \tag{20}$$

The graphical method referred to above, while approximate, is a very powerful method in dealing with irregular shapes because it can be used where mathematical methods fail completely. The graphical method is applied to a numerical case in the example which follows.

Example. Consider a unit axial length L of the inner and outer conductors shown in Fig. 8. This unit axial length multiplied by the area of any one of the three "squares" (areas which face the reader) will be called a "unit volume" for reasons which will become more evident presently. Each of the three "squares" shown between the 30°-displaced flow lines of Fig. 8 are so

[2] The abbreviation ln signifies natural logarithms, and log signifies logarithms to the base 10.

drawn that the average a dimension of a square is equal to the l dimension of a square. If the squares are drawn sufficiently small, the resistance of any unit volume is

$$R_{\text{unit vol.}} = \rho \frac{l''}{a''} = \rho \quad (\text{since } l'' = a'' \text{ by construction}) \tag{21}$$

For the dimensions shown in Fig. 8, there are 3 unit volumes in series between the two conductors and 12 of the series combinations in parallel between the two conductors. Thus the insulation resistance is

$$R_{\text{ins.}} = \frac{3}{12} \frac{\rho}{L} = 0.25 \frac{\rho}{L} \quad \text{ohms} \tag{22}$$

Employing equation (20) to calculate the insulation resistance, there is obtained

$$R_{\text{ins.}} = \frac{\rho}{2\pi L} \ln \frac{2.4}{0.5} = \frac{\rho}{2\pi L} 1.57 = 0.25 \frac{\rho}{L} \quad \text{ohms} \tag{23}$$

In using equations (22) and (23) it is assumed that, if ρ is expressed in ohms per centimeter cube (or ohm-square centimeters/centimeter), L will be expressed in centimeters.

For the case considered above, the graphical method (or the curvilinear-square method as it is usually called) yields the same result (to two significant figures) as does the analytical method. A considerable amount of experience, however, is necessary before the curvilinear-square method can be used with facility in more general cases. This method will be introduced gradually throughout the text wherever it will help to give the reader a better grasp of the physical facts. The mapping of flow (or flux) lines and equipotential surfaces is a valuable adjunct to the analytical approach where electric and magnetic fields are involved.

Equation (20) cannot be used successfully to determine insulation resistance unless the exact value of ρ is known, and even for a specified material this value varies widely with temperature, potential gradient, and moisture content. For example, the ρ of rubber compounds varies widely with the success with which steam is prevented from condensing in the insulation during the vulcanizing process. Numerically, ρ varies from about 10^{10} to 10^{16} ohms/cm cube for good insulators.

11. Resistance Calculation from a Field Map. If a region possesses some physical quantity (at each point in the region) which is a continuous function of the space coordinates, the region is called a *field*. The region between two surfaces which are maintained at a constant temperature difference, for example, is a temperature field because each point in the region can be characterized by the scalar quantity temper-

ature or by the vector quantity temperature gradient. A map of this temperature field would consist of various isotherms (lines or surfaces of equal temperature) and heat-flow lines directed along the lines of maximum temperature gradient.

The three "squares" shown in Fig. 8 constitute a map of the electric field between the two 30°-displaced current lines, and it has already been shown how resistance can be calculated from a map of this kind. It will be observed in Fig. 8 that a "square" is bounded by flow (or current) lines and equipotential lines. Assume, for example, that the inner conductor in Fig. 8 is at a potential of 300 volts relative to the outer conductor which is at zero potential owing to being grounded. The inner boundary of the first square is an equipotential line of 300 volts, and the outer boundary of this square is an equipotential line of 200 volts since

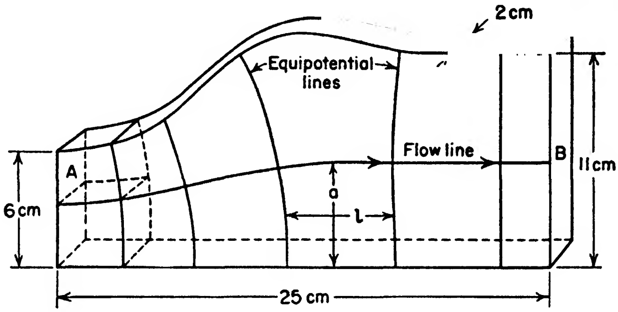


FIG. 9. Irregular-shaped conductor.

⅓ of the total potential drop between the conductors occurs across the l' length, because the "unit volumes" have been so constructed that each one has the same resistance. [See equation (21).] The outer boundary of the second square in Fig. 8 is then at a potential of 100 volts, the outer boundary of the third square being at 0 volts. Maps of this kind are widely used in estimating voltage gradients, a topic which will be considered in some detail in Chapter VII.

By way of further illustration of resistance calculations from field maps, consider the irregular-shaped conductor shown in Fig. 9. If the two surfaces A and B are maintained at a constant potential difference, current will flow along the long dimension of the conductor. Three flow lines are shown in the drafting plane in Fig. 9, the upper and lower boundary of the conductor and a "mid" flow line which divides the conductor into two parallel elements between the A and B faces. After this "mid" flow line is sketched in tentatively, the equipotential lines are drawn in as shown in Fig. 9 in such a way that the average distance between the equipotential lines for any "square" is the same as the

average distance between the flow lines which bound this "square." In using averages in this manner, it is assumed that fairly uniform boundaries of the "squares" exist, as they do in Fig. 9. *Equipotential lines cross flow lines at right angles,* and in curvilinear-square field mapping, *adjacent equipotential lines are separated from one another by the same average distance as the adjacent flow lines are separated from one another.* If these two simple rules are followed a reasonably accurate map (say within 5 per cent) can be sketched in a relatively short period of time where the irregularities are not too pronounced.

After the map has been sketched in, it is a relatively easy matter to find the resistance from surface A to surface B because each "elemental" volume has ρ/z units of resistance. (z is the dimension at right angles to the drafting plane, 2 cm in Fig. 9.) That is,

$$R_{\text{elemental vol.}} = \frac{\rho l}{za} = \frac{\rho}{z} \quad \text{(because } a = l\text{)} \tag{24}$$

Since there are 5.5 elemental volumes in tandem (or series) between faces A and B, the resistance from A to B between two adjacent flow lines is $(5.5\,\rho)/z$. In Fig. 9, there are two such paths (in parallel) joining surfaces A and B; therefore

$$R_{AB} = \frac{5.5\rho}{2z} \tag{25}$$

It will be observed that the horizontal and vertical dimensions shown in Fig. 9 do not enter into the final result. Except for the z dimension (2 cm in Fig. 9), all other dimensions could be changed to any other unit, and the resulting resistance would be the same as for the centimeter units specified, the reason being that the ratio of a to l in the elemental volumes is not changed by a change in the horizontal and vertical units. The mapping process as used here may be interpreted as a graphical technique whereby an irregular-shaped body is divided into elements which are sufficiently small to permit the application of $R = \rho l/A$ to each element.

PROBLEMS

1. If a positive current (one in the $+\mathcal{E}$ direction) of 2 amp is flowing across a cross-sectional area (the face of which is at right angles to the $\mathcal{E}$ vector) and an electron flow of 10×10^{19} electrons/sec is simultaneously crossing this area in the $-\mathcal{E}$ direction, what is the equivalent positive current, $+I$, crossing the area?

2. Two flat metal electrodes are separated from one another by a distance of 0.1 in. If these electrodes are immersed in an insulating oil of the kind specified on page 72, at what potential difference could a flash-over between the electrodes be expected?

3. What is the current density (in amperes per square centimeter) in a wire of rectangular cross section (0.05 in. by 0.20 in.) if a current of 20 amp flows in the wire?

4. A rough working rule which is sometimes employed in small transformer design is to provide 1 circular mil area (of copper) per milliampere of current. What is the allowable current under this rule if No. 18 B. & S. gage wire is used? Express result in amperes.

5. If electrons pass a cross section of a conductor (0.05 in. by 0.20 in.) at the rate of 10×10^{19} electrons/sec, what is the current density in amperes per square inch?

6. If a pessimistic estimate of 1 *free* electron per 3000 atoms of conductor material in Prob. 3 is assumed, what is the velocity of the electron drift? (Assume that there are 8.4×10^{22} atoms/cu cm).

7. In a particular section of conductor ($A \times \Delta l$), say of a gas-filled tube, the negative charge carriers outnumber the positive charge carriers 4 to 1. Each negative charge carrier is an electron (-1.6×10^{-19} coulomb) and is traveling in the $-$ to $+$ direction with a velocity of 10^7 cm/sec. Each positive charge carrier is a doubly charged ion ($+3.2 \times 10^{-19}$ coulomb) and travels in the $+$ to $-$ direction with a velocity of 10^4 cm/sec. If the density of the electrons in the ($A \times \Delta l$) section which is being considered is 10^{10} electrons/cu cm, find the equivalent positive current passing a cross-sectional area of 1 sq cm, the face of which is normal to the $-$ to $+$ direction.

8. What is the voltage drop (at 20°C) along 1000 ft of the wire specified in Prob. 3 if the material is hard-drawn copper?

9. Compare the resistance of 1000 ft of No. 10 B. & S. gage *commercial* annealed copper wire with that given in Appendix A for *standard* annealed copper. [*Standard* annealed copper which is reckoned as 100 per cent conductivity copper has a resistivity (by international agreement) of 1.7241×10^{-6} ohm/cm cube or 10.368 ohms/cir-mil-ft at 20°C.]

10. One hundred per cent conductivity of conducting materials is sometimes reckoned in terms of the International Annealed Copper Standard, which corresponds to 1.7241×10^{-6} ohm/cm cube at 20°C. (This resistivity is slightly lower than that of *commercial* annealed copper.) What is the per cent *conductivity* of hard-drawn copper if its resistivity is 1.7758×10^{-8} ohms/meter cube.

11. What is the resistance of 100 ft of No. 30 B. & S. gage *Advance* wire at 20°C? *Advance* is a trade name for a nickel-copper alloy having 55% copper and 45% nickel. See page 81.

12. What is the resistance of 6000 ft of commercial annealed copper wire (cross section of 0.10 in. by 0.10 in.) at 80°C? What is the resistance under the same conditions if the copper is hard-drawn?

13. A potential drop of 30 volts occurs over a 1000-ft length of conductor of uniform cross section. Find the resistivity of the conductor material in ohms per inch cube if the current density is 2500 amp/sq in.

14. What is the resistivity in ohms per circular-mil-foot of platinum at 60°C?

15. What is the 0°C resistivity temperature coefficient of annealed tungsten?

16. The resistivity of a particular negative-temperature-coefficient resistor is expressed as

$$\rho = 4.54\varepsilon^{2730/T} \text{ ohms/cm cube}$$

where T is in degrees Kelvin and ε is the base of the natural logarithms. What is the 0°C value of resistivity? What is the 0°C temperature coefficient of resistivity?

17. The resistance of a coil of copper wire is known to be 10 ohms at 20°C. With a voltage of 10 volts applied to the terminals of the coil, the current finally reaches an ultimate value of 0.928 amp. What is the final temperature of the coil?

18. What is the temperature of a nickel wire (0.01 cm by 0.01 cm in cross section and 10 cm long) if its resistance is 1.012 ohms?

19. What is the insulation resistance of 10 miles of coaxial cable, the inner conductor of which is 0.10 in. in diameter and the inside diameter of the outer conductor of which is 1.0 in.? The resistivity of the insulation material is 6.28×10^{12} ohms/cm cube, and this resistivity is assumed to be independent of the potential gradient.

20. What is the approximate resistance of the irregular-shaped conductor shown in Fig. 9, page 87, if the material is a nickel-chromium alloy composed of 80% nickel and 20% chromium? Express result in microhms.

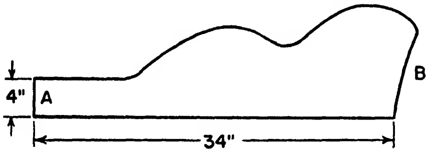


FIG. 10. See Prob. 21.

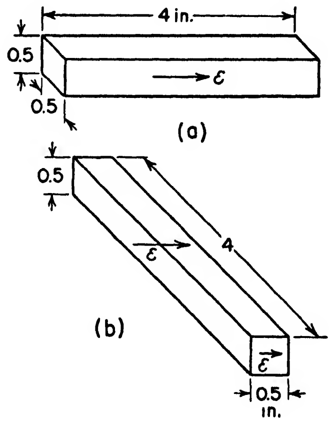


FIG. 11. See Prob. 22.

21. The irregular-shaped conductor shown in Fig. 10 is a thin sheet of nickel-chromium alloy which has a dimension of 0.1 in. into the page. What is the approximate resistance, expressed in microhms, of this conductor from the A end to the B end? $\rho = 100$ microhm-sq cm/cm.

22. Refer to Fig. 11. In (a) the potential difference is so applied that the $\mathcal{E}$ vector is along the 4-in. dimension as shown. In (b) the potential difference is applied to the same conductor so that the $\mathcal{E}$ vector is along one of the 0.5-in. dimensions. What is the ratio of the resistance in (a) to that in (b)?

23. What is the current density (in amperes per square centimeter) of a fine beam of electrons which has a volume density of 10^8 electrons/cu cm if the electrons are traveling at a velocity of 6.25×10^8 cm/sec?

24. The conductor shown in Fig. 12-a is made of Nichrome which has a resistivity of 100×10^{-6} ohm/cm cube and has the dimensions given in the figure.

(a) Determine the resistance between surfaces A and B employing the field map given in Fig. 12-b.

(*b*) Determine the resistance between surfaces *A* and *B* on the assumption of parallel flow lines in the large section up to the reduced section and again parallel flow lines in the reduced section. Compare this value of resistance with that obtained from the field map.

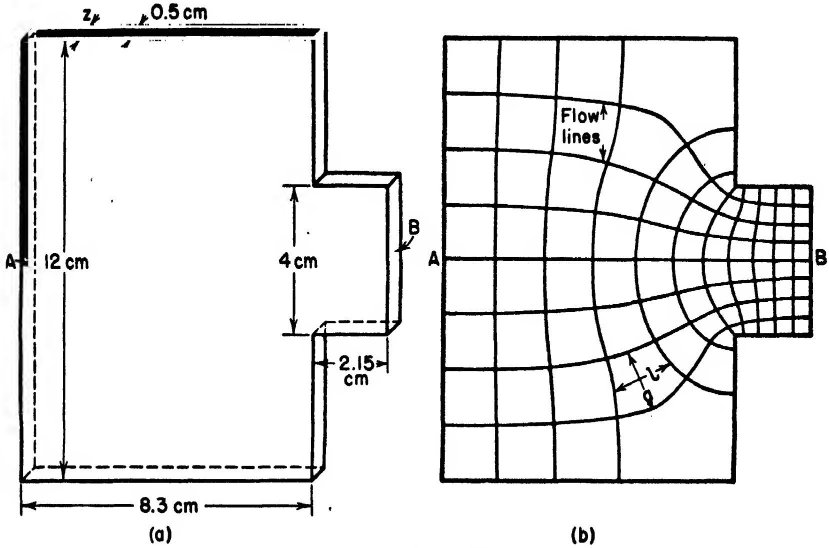


FIG. 12. See Prob. 24.

25. (*a*) What is the conductivity of the insulation material described in Prob. 19 expressed in mhos per meter cube?

(*b*) What is the conductance (conductor-to-conductor) of the coaxial cable expressed in mhos per mile?

26. Find the resistance between the parallel faces of a conductor, the shape of which is the frustum of a right circular cone, if the parallel circular faces have radii of 1 cm and 2 cm, respectively, and the axial length of the conductor is 20 cm. The resistivity of the material is $\pi/1000$ ohms/cm cube.

CHAPTER V

Joule's Law — Heating Effects

1. Heat Energy. Although we have described *resistance* as that property of an electrical conductor which causes an irreversible transformation of electrical energy to heat energy, we have not described the mechanism whereby this transformation of energy takes place.

At temperatures above absolute zero (−273°C), the constituent particles (of which matter is composed) possess energy due to their individual movements or vibrations. These individual movements are restricted or limited by the internal binding forces which hold the particles together to form solids, liquids, or gases. Heat energy represents the sum of the kinetic energies of these constituent particles. The higher the temperature of the matter the greater is the random movement of the constituent particles within the limits permitted by the internal binding forces. If this random movement is increased (say, in a current-carrying copper conductor) to a point where the internal binding forces can no longer maintain the crystalline structure of solid copper, the copper melts with obvious detrimental effects to the electrical circuit.

The conduction of electrical charge through solid, liquid, or gaseous conductors[1] increases the kinetic energy of the constitutent particles of the conductor, and the increase in temperature which results actually limits the available output of most electrical apparatus.

2. Heating Effect of Electron Drift. The conduction of electric charge through metals consists of a net (or time-averaged) movement of electrons in the

$-$ to $+$ direction through dissipative circuits
$+$ to $-$ direction through electrical generators

Except in generators where outside energy is imparted to the electrons to drive them in the $+$ to $-$ direction, the electrons tend to move in the $-\mathcal{E}$ direction, that is, in the direction of increasing voltage rise, $+dE/dl$.

[1] Since the passage of electrical charge across the boundaries between certain dissimilar conductors can actually lower the temperature of the material, we consider here only homogeneous conductors. Boundary phenomenon is a complicated subject which will be considered briefly in a later chapter.

In Fig. 1 is shown an elementary circuit in which the metallic conductor which joins the − and + terminals of the generator is so drawn that the reader may partially visualize the actual drift of *free electrons* within the conductor. Although the atoms which furnish the free electrons are left with a net positive charge, they do not contribute to the net movement of electric charge around the circuit because in the solid state these atoms are held together by atomic forces or bonds which prevent any net movement of the nuclei.

The free electrons, however, move through the spaces within the conductor when a potential gradient $(-\mathcal{E})$ is present. Their movement through the conductor is hampered to some extent by collisions with atoms, or, more precisely, by their coming within the sphere of influence of neutral atoms, the outer shell of which is negatively charged. (This opposition to the net movement of electrons is what we have previously called *resistance.*) When a moving electron approaches a neutral atom more or less head on, it encounters a repulsion effect due to the orbital electrons which surround the nucleus of the atom. The kinetic energy possessed by the moving electron is then wholly or partially transferred to the atom, and this energy serves to increase the thermal energy of the atom. The temperature of the conductor is therefore increased as a result of current flow within the conductor. This phenomenon constitutes the basis of operation of heating devices like electric stoves, electric furnaces, and cathode heaters in vacuum tubes.

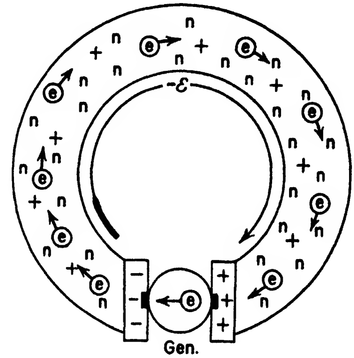


FIG. 1. Illustrating the haphazard movements of electrons e in a metallic conductor. The n's indicate neutral atoms.

The movement of any one electron (say electron A) in the conduction process is a complicated and erratic movement because this electron may, as a result of its kinetic energy, enter the outermost energy level of an otherwise normal atom. This atom may then release a different electron (say electron B) from its outermost energy level, and the latter electron carries on the conduction process where electron A left off. In spite of the haphazard movement of any single electron, the net movement of charge is to all outward appearances a very uniform process because so many billions of individual electrons are involved.

The average rate at which the moving electrons deliver kinetic (or thermal) energy to the atomic structures of the conductor is directly proportional to the square of the average net velocity attained by the

electrons. This velocity is directly proportional to the current density since

$$J \text{ (current density)} = Nqv \qquad \text{(See page 76.)}$$

Hence the rate at which heat is developed within a specified conductor is directly proportional to the square of the current which is flowing in the conductor.

3. Joule's Law. Joule, about 1841, verified the fact experimentally that the rate at which heat is developed in a uniform conductor is directly proportional to the square of the current and directly proportional to the resistance of the conductor. This fact is known as Joule's law, and it may be expressed in equation form as

$$p \text{ (rate at which heat is developed)} = \frac{dH}{dt} = Ri^2 \tag{1}$$

where H is used to symbolize heat energy. Both p and i are written in lower-case letters here to indicate that instantaneous values of power and current are intended.

As applied to a continuous current I flowing through a dissipative resistor R, equation (1) may be written:

$$P = VI = \frac{V^2}{R} = RI^2 \tag{2}$$

because in this case all the electric power VI which is delivered to the resistor R is transformed irreversibly into heat. Volts, amperes, and ohms are customarily used in equation (2), in which case P is given in joules per second or watts.

4. Heat Equivalent of Electrical Energy. Heat energy is often expressed in gram-calories, the heat required to raise the temperature of 1 g of water 1°C; or in Btu, the heat required to raise the temperature of 1 lb of water 1°F. In order to find the relation between a joule of electrical energy and a gram-calorie of heat energy, it is simply necessary to supply a specified (or known) number of grams of water with a known number of joules of electrical energy and observe the temperature rise of the water. This may be done by immersing a dissipative resistor in a thermally insulated calorimeter containing the water and arranging to measure

$$\text{electrical energy} = VIT \quad \text{joules} \tag{3}$$

where V is the voltage drop across the resistor in volts, I is the current flowing through the resistor in amperes, and T is the time in seconds. The arrangement is shown schematically in Fig. 2 and the test here described is often included in the list of first-course physics experiments.

After an appropriate period of time T, the temperature rise of the water is measured, and from this

$$\text{heat energy} = mT^\circ \quad \text{g-cal} \tag{4}$$

where m is the number of grams of water (including the water equivalent of the thermometer, stirrer, and other apparatus immersed in the water) and T° represents the temperature rise in centigrade degrees.

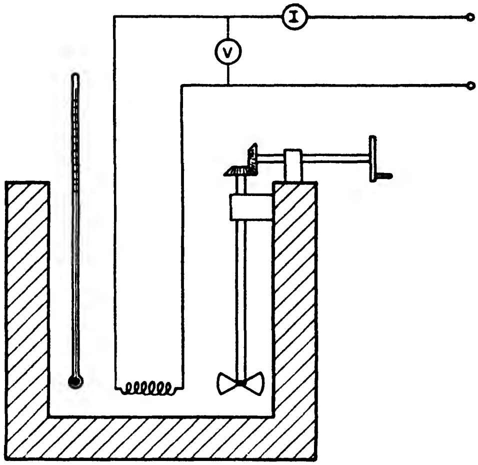


FIG. 2. Schematic arrangement of apparatus for measuring the electrical equivalent of heat, g-cal/watt-sec or g-cal/joule.

If careful measurements are made, the ratio of the result obtained in (4) to that obtained in (3) is

$$\frac{\text{g-cal}}{\text{joule}} = 0.2388 \tag{5}$$

which means that the *number* of gram-calories per joule is 0.2388. This relationship may be expressed as

$$1 \text{ g-cal} \equiv 4.186 \text{ joule} \equiv 4.186 \text{ watt-sec} \tag{5-a}$$

Thus it may be seen that the gram-calorie is a unit of energy which is 4.186 times as large as the joule. In equational form

$$(\text{No. of}) \text{ g-cal} = 0.2388\, VIT \tag{6}$$

or

$$(\text{No. of}) \text{ g-cal} = 0.2388 \frac{V^2}{R} T = 0.2388 RI^2 T \tag{6-a}$$

where V is in volts, I is in amperes, R is in ohms, and T is in seconds.

Example. Ten gallons of water are to be heated from a feed-water temperature of 50°F to a temperature of 140°F electrically with a heating system which is 0.9 efficient. Find the cost of the electrical energy required to heat the water if electrical energy can be purchased for $0.03 per kwhr. (The assumption will be made that 1 gal of water is equivalent to 3785 g of water irrespective of its temperature.)

$$\text{useful heat energy required} = 10 \times 3785 \times \tfrac{5}{9}(140 - 50) = 1.89 \times 10^6 \text{ g-cal}$$

$$\text{input energy from electrical source} = \frac{1.89 \times 10^6}{0.9} = 2.1 \times 10^6 \text{ g-cal}$$

$$\text{input energy in joules or watt-seconds} = (2.1 \times 10^6) \times 4.186 = 8.79 \times 10^6$$

$$\text{input energy in kilowatt-hours} = \frac{8.79 \times 10^6}{3.6 \times 10^6} = 2.44$$

$$\text{cost of energy to heat 10 gal of water} = 2.44 \times 0.03 = \$0.0732$$

5. Ultimate Temperature Rise in Electrical Apparatus. The useful output of most electrical apparatus is limited by the ultimate temperature rise of the current-carrying conductors. Many electrical machines have current-carrying conductors which are insulated with organic materials which can withstand ultimate temperatures up to 90° or 95°C only. If machines of this kind are to operate at ambient temperatures of 40°C, they are designed to operate at full load with ultimate temperature rises of 50° or 55°C. If glass, mica, or other inorganic insulating materials are employed much higher temperature rises can be tolerated.

The ultimate temperature rise of electrical apparatus is reached when the rate at which heat is developed equals the rate at which heat is transferred to the surrounding atmosphere. The rate at which heat is developed within the apparatus can usually be determined with accuracy. The rate at which heat is transferred to the surrounding atmosphere depends upon the thermal conductance of the various parallel heat paths by which heat escapes to the surrounding atmosphere. Since these paths are usually of irregular geometrical shape and since the actual transfer of heat is dependent upon convection currents, the problem of finding the rate at which heat is transferred from the apparatus is a complicated one.

Some aspects of this complicated heat-flow problem may be understood by considering a simple two-dimensional heat flow situation in which

$$\text{rate of heat transfer by conduction, } \frac{dH}{dt} = CT^\circ = \frac{cA}{l}T^\circ \qquad (7)^2$$

[2] The correspondence between $dH/dt = CT^\circ$ and $dQ/dt = I = GV$ is an analogy which can sometimes be used to advantage. The time rate of heat transfer (by con-

where dH/dt is usually expressed in gram-calories per second or in joules per second (watts)

$C \left(= \dfrac{cA}{l}\right)$ is the thermal conductance of the heat path

c is the thermal conductivity of the material through which the heat is transferred

A is the cross-sectional area of the path

l is the length of the path in the direction of the flow lines

$T°$ is the temperature difference between the ends of the path.

If A and l in equation (7) are expressed in square centimeters and centimeters respectively and $T°$ is expressed in centigrade degrees, c will be expressed in $\dfrac{\text{g-cal} \times \text{cm}}{\text{sec} \times \text{sq cm} \times °\text{C}}$ if the result is to be in gram-calories per second. This unit of c is often tabulated in handbooks for various materials as g-cal/sec/cm^3/°C. Copper, for example, has a thermal conductivity of 0.918 g-cal/sec/cm^3/°C, and vulcanized rubber has a thermal conductivity of about 0.0004 g-cal/sec/cm^3/°C.

If dH/dt is to be expressed in joules per second or watts, we might care to express c in (watts × cm)/(sq cm × °C) or in watts/cm^3/°C. The value of c in these units would be 4.186 times as large as c expressed in g-cal/sec/cm^3/°C because of the relative size of the gram-calorie and the joule.

Example. Consider a 100-cm length (into the page) of the coaxial cable shown in Fig. 8, page 84, where the inner surface of the outer conductor is assumed to be at a fixed temperature of 30°C. The electrical insulation between conductors is assumed to be vulcanized rubber having a thermal conductivity of 0.0004 g-cal/sec/cm^3/°C. What is the ultimate temperature rise of the inner conductor (above 30°C) if 20 watts of electrical power are dissipated continuously in the inner conductor?

As in most practical situations involving heat transfer, the cross-sectional area of the heat path in Fig. 8, page 84, is a function of the length of the path. Since the geometrical configuration of the heat path is well defined in this case, an analytical solution is possible. The analytical method will be employed here, and later the same problem will be solved by a graphical method.

duction) is governed by the product of the thermal conductance and temperature difference, whereas the time rate of electric charge is governed by the product of electrical conductance and potential difference.

We note first that for any differential length of path, dx, the corresponding area is $(100)(2\pi x)$ or $200\pi x$ sq cm. The differential thermal resistance of the dx length is

$$d\left[\frac{1}{C}\right] = d\left[\frac{dx}{c(200\pi x)}\right]$$

Integrating both sides of the above equation,

$$\frac{1}{C} = \frac{1}{c}\int_{0.5}^{2.4} \frac{dx}{(200\pi)x} = \frac{1}{0.2512} \ln \frac{2.4}{0.5} = 3.98 \times 1.57 = 6.24$$

or

$$C = \frac{1}{6.24} = 0.16 \quad \text{g-cal/sec/°C}$$

Since, as a ratio, joules/g-cal = 4.186, C may be expressed in watts per centigrade degree as

$$\frac{\dfrac{\cancel{\text{g-cal}}}{\text{sec}} \times \dfrac{\text{joules}}{\cancel{\text{g-cal}}}}{\text{°C}} = \frac{0.16 \times 4.186}{\text{°C}} = 0.67 \text{ watts/°C}$$

Hence the temperature difference between the end surfaces of the heat path is

$$T° = \frac{\text{watts}}{\dfrac{\text{watts}}{\text{°C}}} = \frac{20}{0.67} \doteq 30\text{°C}$$

If it be assumed that the temperature difference within the inner copper conductor is negligibly small as compared with the drop in temperature in the rubber insulation, we may say that the surface of the inner conductor is at a temperature of 30° + 30° or 60°C.

Although the analytical method of calculating temperature difference is useful in cases where the geometry is simple, it is of little use in those cases where the *area* cannot be represented analytically and as a manageable function of the *length* of the heat path. Complicated geometrical shapes are the rule rather than the exception in most of the heat transfer problems encountered by the electrical engineer. Field mapping is, therefore, resorted to in finding approximate solutions to heat conduction problems in irregular-shaped bodies. An example of this method is given below.

Field Mapping Method. The field map drawn in Fig. 8, page 84, can be used to advantage here as a temperature field map simply by invoking the analogy referred to in footnote 2.

We know that the inner surface of the outer conductor is an equi-temperature surface (30°C as specified) and that the conductance of an elemental volume is

$$0.0004 \text{ g-cal/sec/°C} \quad \text{or} \quad 0.00167 \quad \text{watts/°C}$$

because the curvilinear squares of the field map have been so constructed that a/l of each square is unity. In this connection it will be remembered that an elemental volume is the volume represented by any square (as indicated in Fig. 8, page 84) times 1-cm length into the plane of the page.

The thermal conductance of a 100-cm length of insulation (between inner and outer conductor) is, from the graphical construction shown in Fig. 8,

$$C = 0.00167 \times \tfrac{1}{3} \times 12 \times 100 = 0.67 \quad \text{watts/°C}$$

since there are 3 elemental volumes in series and 1200 in parallel between the two conductors.

The 20 watts of power developed within the inner conductor is the dH/dt member of equation (7); therefore, considering only the simple

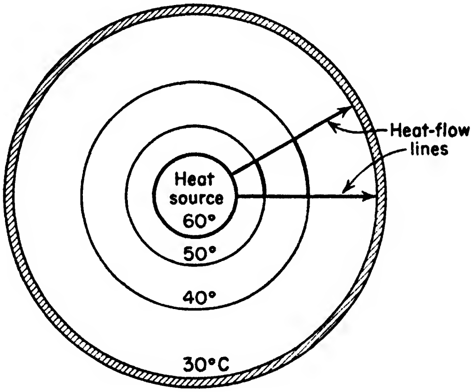


FIG. 3. Cross-sections of equitemperature surfaces.

conduction path through the rubber insulation, we write, from equation (7),

$$T^\circ \text{ (rise above 30°C)} = \frac{\frac{dH}{dt}}{C} = \frac{20}{0.67} \doteq 30\text{°C}$$

From the field map shown in Fig. 8, page 84, the temperature at various places within the insulation material can be determined with a fair degree of accuracy. Starting with the inner surface of the outer conductor which has been specified as being at 30°C, the 40°C, 50°C, and 60°C equitemperature surfaces are defined by the circular arcs which bound the squares of the field map. Cross-sectional views of these equitemperature surfaces are shown in Fig. 3. Field maps of this kind

are sometimes employed by designers to approximate the internal temperatures in electrical apparatus.

6. Watts per Square Inch as an Index to Temperature Rise. The determination of ultimate temperature rise is not actually as simple as indicated in the above example because the outer surface of the apparatus is usually exposed to air. Instead of the outer surface being held at a constant temperature as specified in the example, the heat energy is transferred from the outer surface by radiation and convection. Transfers of heat by radiation and convection do not follow the simple linear relationship given in equation (7), so recourse must be made to empirical data in the actual determination of temperature rise. Equation (7) can be rearranged to show that

$$\text{temperature rise } T^\circ = k' \times \left[\frac{\text{watts}}{\text{sq in.}}\right] \tag{8}$$

where k' is dimensionally degrees/watts/sq in.

If k' were a constant (for any particular geometrical configuration of a specified material), it could be evaluated experimentally and then used to predict T° because the watts to be dissipated are usually known and the square inches of radiating surface can be calculated. Unfortunately, k' is not a constant because T° does not vary linearly with watts per square inch owing principally to the non-linearity of heat transfer by convection and radiation. Even though considerable judgment must be employed in the use of k' as a means of determining ultimate temperature rise, designers frequently employ this method. In the final analysis, the method involves knowing from past experience the temperature rise which is produced in a given geometrical configuration for *some* number of watts per square inch which is reasonably close to the actual number of watts per square inch for which the apparatus is designed to operate.

Small compact resistors where no part of the material is more than about ⅛ in. from a radiating surface usually reach their allowable working temperatures when subjected to roughly 1 watt/sq in. of radiating surface. Where resistance wire is mounted on the surface of porcelain tubing or the like, relatively high working temperatures can be safely employed, and in these cases only ½ or ⅓ of a square inch of radiating surface is provided for each watt of power to be dissipated. Where the current-carrying conductors are deeply imbedded (say an inch or more from the radiating surface) it is sometimes necessary to provide 5 or more square inches of radiating surface per watt in order to keep the internal temperature rise to its proper value.

That the geometrical configuration is of great importance in deter-

mining ultimate temperature rise may be seen by observing factual information in a particular case:

(*a*) No. 14 B. & S. gage rubber-covered copper wire when strung out in air can safely carry 15 amp without overheating.

(*b*) The same wire wound into a 1-ft diameter roll of one or two hundred turns carrying 5 amp will be dangerously overheated.

Another example might be based on the configuration shown in Fig. 3 if we consider the effect of reducing the thickness of insulation from that shown. If the outer conductor actually occupied the 50° cross-section shown in Fig. 3 and if this conductor were held to 30°C as previously specified, the outer surface of the inner conductor would rise to only 40°C as contrasted to the 60°C temperature attained when the thicker insulation was employed.

Example. Let it be required to find the (temperature rise/watt/sq in.) of a small coil of copper wire [k' of equation (8)] from the following data:

1. The coil has a radiating surface of 10 sq in.
2. At 20°C room temperature, the resistance R_1 is known to be 10 ohms.
3. With a constant potential difference of 10 volts across the terminals of the coil, the current finally reaches an ultimate value of 0.9 amp.

The space-averaged internal temperature of the coil with 0.9 amp flowing continuously may be determined with the aid of equation (17), page 83, as

$$T_2 = \frac{R_2}{R_1}(234.5 + T_1) - 234.5$$

$$= \frac{11.11}{10}(234.5 + 20) - 234.5 = 48°\text{C}$$

Watts per square inch of radiating surface:

$$\frac{\text{watts}}{\text{sq in.}} = \frac{10 \times 0.90}{10} = 0.90 \quad \text{(for 48°C rise)}$$

$$k' = \frac{48}{0.90} = 52.8°\text{C} \quad \text{rise/watt/sq in.}$$

Because T_2 has been evaluated as a space-averaged temperature, it is to be expected that somewhat higher temperatures than $T_2 = 48°\text{C}$ will exist at those points within the coil which have the greatest thermal insulation. This increase might be 5° or 10°C, depending upon the exact geometrical configuration of the coil.

7. Effective or Ampere Value of a Time-Varying Current. In many cases the current in a circuit is not a constant; that is, dq/dt across a cross-sectional area of the circuit is not constant. If the current varies

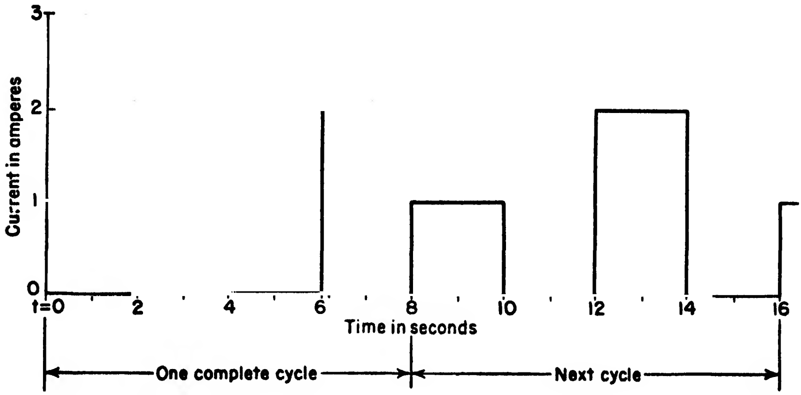


FIG. 4. A cyclic current variation produced by a rotary contactor like that shown in Fig. 5.

with time, the *instantaneous* value varies from instant to instant, and the question of the ampere value of this current variation naturally arises. For the present, only cyclic current variations will be considered; that is, the current after varying in some prescribed manner for a period

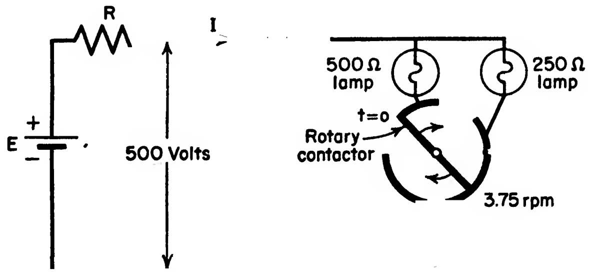


FIG. 5. I varies alternately between 1 amp, zero, 2 amp, and zero. ($R \ll 250$ ohms.)

of time repeats precisely the same cycle of variation, cycle after cycle. A cyclic variation of current is shown in Fig. 4, and a circuit arrangement by means of which the current cycle can be produced is indicated in Fig. 5.

The ampere or effective value of a time-varying current is defined in such a way that, ampere for ampere, the time-varying current produces the same heat energy during the time of one cycle as would the same number of amperes of steady or direct current.

It has been shown that the time rate of heat production in a resistive circuit is

$$p = \frac{dH}{dt} = Ri^2 \quad \text{joules/sec} \quad \text{or} \quad \text{watts} \tag{9}$$

in which H is employed as a symbol for heat energy.

Counting the beginning of a cycle of current variation as $t = 0$ and the length of time of a complete cycle as T, we find from equation (9) by separating variables and integrating that

$$H = \int_0^T Ri^2\,dt \quad \text{joules} \quad \text{or} \quad \text{watt-sec} \tag{10}$$

The heat energy produced by a cyclic current of the kind shown in Fig. 4 over the period of time of one complete cycle is

$$H = \int_0^2 R \times 1^2\,dt + \int_2^4 R \times 0\,dt + \int_4^6 R \times 2^2\,dt + \int_6^8 R \times 0\,dt \qquad (11)^3$$

in which the intervals between 2 sec and 4 sec and between 6 sec and 8 sec contribute nothing since $i = 0$ in these intervals.

The heat energy developed in the resistance R due to one cycle of current shown in Fig. 4 is found from equation (11) to be

$$H = 10R \quad \text{joules or watt-sec} \tag{12}$$

The resistance R is any fixed resistance through which the current in question might be flowing for purposes of this analysis.

The heat energy produced by a steady or direct current flowing through the same resistance R for a period of 8 sec would be

$$H_{dc} = RI^2T = 8RI^2 \tag{13}$$

where I is the equivalent direct current.

If the effective or ampere value of the time-varying current shown in Fig. 4 is defined as the same as an equivalent direct current which will produce the same heating effect, $8RI^2$ [from equation (13)] may be equated to $10R$ [from equation (12)] and

$$I^2 = \tfrac{10}{8} \quad \text{or} \quad I = 1.12 \quad \text{amp}$$

In other words, 1.12 amp of direct current will produce the same heating effect in any resistance as would the time-varying current shown in Fig. 4.

[3] Equation (11) is simply a formal way of summing up the total energy developed by the current variation shown in Fig. 4. The actual sum in this simple case is plainly ($2R1^2 + 2R2^2 = 10R$ watt-sec).

In general, the effective value of a time-varying current i is defined as

$$I_{\text{eff.}} = \sqrt{\frac{1}{T}\int_0^T i^2\,dt} \qquad (14)$$

where the integration indicated is performed over one complete cycle, from 0 to T. In some types of current-time graphs, the integration need be carried out only from 0 to $T/2$ or from 0 to $T/4$ depending upon the type of symmetry involved. A little study of the expression under the radical sign in equation (14) will show that it is the *average* squared value of the instantaneous current over one cycle. Since the effective current is the square root of the average squared value of current, it may be evaluated either by analytical integration or by any graphical method which is applicable to finding the time-averaged squared value of the current throughout one cycle. After finding the time-averaged squared value of current, it is simply necessary to extract the square root in order to find the *effective* value of current in accordance with the general definition of effective current which has been given in equation (14).

In finding the effective value of the time-varying current shown in Fig. 4, for example, one might divide the cycle into four equal time intervals, each of 2 sec duration, and proceed to evaluate the square root of the average squared value of current on this graphical basis as

$$I_{\text{eff.}} = \sqrt{\frac{1+0+4+0}{4}} = 1.12 \quad \text{amp}$$

In this case the precise value of $I_{\text{eff.}}$ ($\sqrt{5/4}$ amp) is obtained by the graphical method, but in general the precise value can only be approximated by the graphical method. The precise value may, however, be approached as closely as we choose if we select a sufficiently large number of incremental time intervals upon which to base the graphical computations.

If i in equation (14) can be so expressed as a function of time that ($i^2\,dt$) can be integrated formally, graphical methods which are often tedious can be avoided.

Example 1. Applying equation (14) to the current-time graph shown in Fig. 4, one obtains

$$I_{\text{eff.}} = \sqrt{\frac{1}{8}\left[\int_0^2 1\,dt + \int_4^6 2^2\,dt\right]} = \sqrt{\frac{10}{8}} = 1.12 \quad \text{amp}$$

In this case the current-time graph is integrated piece-wise, and the pieces between $t = 2$ and $t = 4$ sec and between $t = 6$ and $t = 8$ sec are neglected because $i = 0$ in these intervals.

Example 2. A sinusoidal time variation of current as shown in Fig. 6 is written analytically as

$$i = I_m \sin \omega t$$

in which i is the instantaneous value of current at any time t
I_m is the maximum value of the sinusoidal variation
ω is the angular velocity or angular frequency of the sinusoidal variation expressed in radians per second ($\omega = 2\pi/T$)
t is time in seconds after the beginning of a cycle
T is the time of one cycle as indicated in Fig. 6.

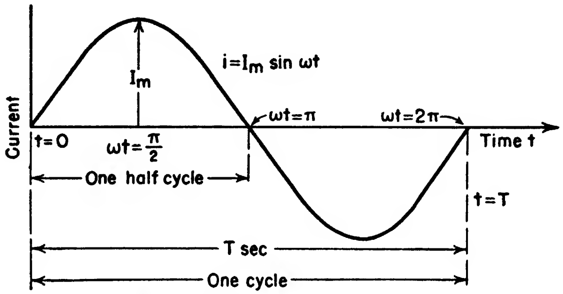


FIG. 6. Illustrating one cycle of a sinusoidal current-time variation.

Applying equation (14) to this type of current variation, one obtains

$$I_{\text{eff.}} = \sqrt{\frac{1}{T}\int_0^T I_m^2 \sin^2 \omega t \, dt} = \sqrt{\frac{I_m^2}{T}\int_0^T \left(\tfrac{1}{2} - \tfrac{1}{2}\cos 2\omega t\right) dt}$$

or

$$I_{\text{eff.}} = I_m \sqrt{\frac{1}{T}\left[\frac{t}{2} - \frac{T}{8\pi}\sin\frac{4\pi}{T}t\right]_0^T} = \frac{I_m}{\sqrt{2}} = 0.707 I_m$$

Thus it is shown that the effective or ampere value of a sinusoidal variation of current is 0.707 times the maximum value.

Example 3. In some cases the analytical expression for i is not known, so graphical integration must be performed to find $I_{\text{eff.}}$. The current-time graph must, of course, be available, and in this particular case we shall assume that the sinusoidal current-time graph shown in Fig. 6 is available but that we choose to evaluate $I_{\text{eff.}}$ by the graphical method rather than by the analytical method given above.

Graphical integration in any case involves merely the evaluation of the square root of the *average squared value* of current over a cycle in accordance with the definition of $I_{\text{eff.}}$ given in equation (14). Owing to the symmetry of the current variation in the present case, the average squared value over one-fourth cycle (from $\omega t = 0$ to $\omega t = \pi/2$) will be the same as the average squared value of current taken over a complete cycle. Some labor may, therefore, be saved by performing the graphical integration over a quarter cycle only.

In this particular case we shall divide the first quarter cycle ($\pi/2$ radians) into 9 equal divisions of 10° each and take as the average value of i in each of these divisions the mid-ordinate, that is, the value of i at 5°, 15°, 25°, and so on. The work involved in finding the average squared value of i is shown below in tabular form.

Interval	$i_{av.}$	$i_{av.}^2$	Interval	$i_{av.}$	$i_{av.}^2$
0°–10°	$0.087I_m$	$0.0076I_m^2$	50°–60°	$0.819I_m$	$0.670I_m^2$
10°–20°	$0.259I_m$	$0.0671I_m^2$	60°–70°	$0.906I_m$	$0.820I_m^2$
20°–30°	$0.423I_m$	$0.1790I_m^2$	70°–80°	$0.966I_m$	$0.934I_m^2$
30°–40°	$0.534I_m$	$0.285I_m^2$	80°–90°	$0.996I_m$	$0.992I_m^2$
40°–50°	$0.707I_m$	$0.500I_m^2$			$\sum i_{av.}^2 = 4.455I_m^2$

Since 9 intervals or divisions have been used in the evaluation of $\sum i_{av.}^2$, $I_{eff.}$ is determined as

$$I_{eff.} = \sqrt{\frac{4.455I_m^2}{9}} = 0.704I_m$$

The correct result is $0.707I_m$ as obtained in Example 2. The result obtained here differs from the correct result by about ½ per cent owing to the finite

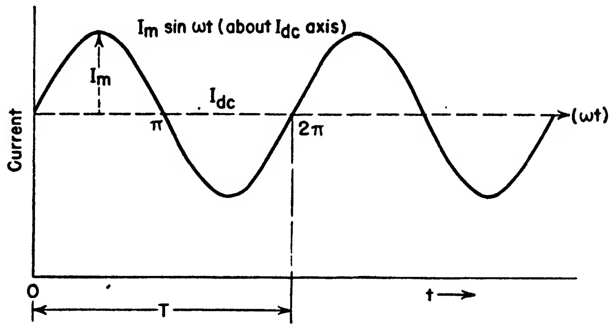


FIG. 7. A sine wave component of current superimposed on a d-c component.

intervals employed. The graphical method can be made more accurate by choosing smaller intervals but, unless the current-time graph is known very accurately, there is little point in trying to obtain accuracies greater than 1 per cent by the graphical method.

Example 4. A type of current variation which is often encountered in vacuum-tube rectifier and amplifier circuits is shown in Fig. 7. As a first approximation, this type of current-time graph may be represented by

$$i = I_{dc} + I_m \sin \omega t$$

in which i is the instantaneous value of current
I_{dc} is the d-c component of current, independent of time t
$I_m \sin \omega t$ is the a-c component of current (see Example 2).

In an amplifier circuit, for example, I_{dc} is required to energize the vacuum tube and $I_m \sin \omega t$ is the useful component of the current.

The effective value of the current variation shown in Fig. 7 is, by equation (14),

$$I_{\text{eff.}} = \sqrt{\frac{1}{T}\int_0^T (I_{dc} + I_m \sin \omega t)^2\, dt}$$

$$= \sqrt{\frac{1}{T}\int_0^T (I_{dc}^2 + 2I_{dc}I_m \sin \omega t + I_m^2 \sin^2 \omega t)\, dt}$$

$$= \sqrt{I_{dc}^2 + \frac{I_m^2}{2}}$$

since

$$\int_0^T I_{dc}^2\, dt = I_{dc}^2 T$$

$$\int_0^T 2I_{dc}I_m \sin \omega t\, dt = 0,$$

and

$$\int_0^T I_m^2 \sin^2 \omega t\, dt = \frac{I_m^2 T}{2}$$

as shown in Example 2.

8. Average Value of Current. The time-averaged value of a time-varying current is

$$I_{\text{av.}} = \frac{1}{T'}\int_0^{T'} i\, dt \tag{15}$$

where T' is the period of time over which the time-averaged value of current is to be found.

If the analytical expression for i is known as a simple manageable function of time, the average value may be found straightforwardly as indicated in equation (15). Otherwise some graphical method like measuring equally spaced ordinates and averaging them may be employed to find $I_{\text{av.}}$.

The average value of a time-varying current is seldom used because it cannot be associated with either R or V to give power as can the effective value of current. In processes like electroplating, where the net total charge flowing through the circuit over a period of time is of importance, $I_{\text{av.}}$ is sometimes used.

As applied to an a-c variation like that shown in Fig. 6, $I_{\text{av.}} = 0$ if T' of equation (15) is taken as the time of a complete cycle. Therefore in an a-c variation which is symmetrical about the zero axis, the average value over one-half cycle is usually taken unless otherwise specified.

Examples. $I_{av.}$ of the time-varying current in Fig. 4 is plainly $\frac{3}{4}$ amp.

The time-averaged value of the current-time graph shown in Fig. 7 is plainly I_{dc} if the period of time considered is an integral number of cycles.

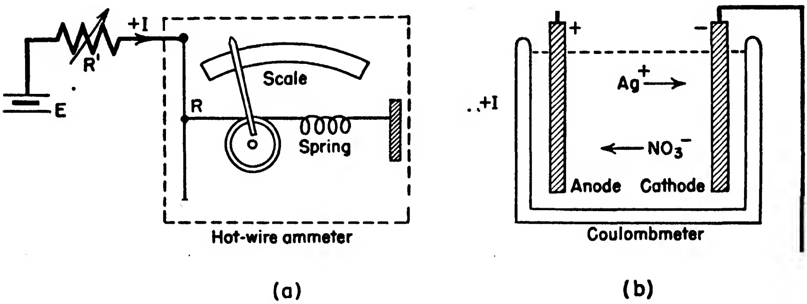
The time-averaged-value of the current-time graph shown in Fig. 6 when taken over one-half cycle, that is from 0 to $T/2$, is

$$I_{av.} = \frac{2}{T}\int_0^{T/2} I_m \sin \omega t \, dt = \frac{2I_m}{T}\left[-\frac{T}{2\pi}\cos\frac{2\pi}{T}t\right]_0^{T/2}$$

$$= \frac{2}{\pi} I_m = 0.637 I_m$$

As in Example 2, page 105, $\omega = 2\pi/T$ is a constant which defines the angular frequency of the a-c variation. If $T = 0.01$ sec, $\omega = 200\pi$ rad/sec. The actual frequency of the variation expressed in cycles per second would be 100 since frequency $= 1/T$.

9. Quantitative Measurement of Current. A hot-wire ammeter[4] can be calibrated with direct or steady current as shown in Fig. 8 and then



(a) (b)

FIG. 8. Hot-wire ammeter in series with a coulombmeter, the latter being used to calibrate the ammeter.

used to measure the effective value of current in other circuits regardless of the wave form of the current. The current which is to be measured heats the resistance wire (R of Fig. 8-a) and the thermal expansion causes elongation of the wire. This elongation is transmitted to a pointer by a suitable mechanical linkage so the scale reading of the pointer is dependent upon the heating effect produced in the wire by the current to be measured.

In the calibration of the hot-wire ammeter, a steady or direct current

[4] Although the hot-wire instrument is of theoretical interest at this point, it is not widely used in practice in the United States. American manufacturers have developed other types of instruments to a much higher degree of accuracy and reliability than they have the hot-wire type of instrument. The commonly used instruments embody magnetic effects which will be considered in a later chapter.

is passed through an electrolytic solution of silver nitrate ($AgNO_3$) which is in series with the hot-wire instrument as shown in Fig. 8. If the pointer of the instrument is held at a marked position on the scale over a period of time (say for 30 min by adjustments of R'), this marked position can be calibrated as a fixed number of amperes. This number of amperes is determined indirectly by weighing the cathode at the beginning of the test and again at the close of the test. The mass of silver deposited on the cathode is directly proportional to Q $(= IT)$ which passes through the ammeter.

Each ion of silver (Ag^+) in Fig. 8–b which is deposited (or plated) on the cathode of the coulombmeter carries 1.6×10^{-19} coulomb of charge since it is a singly charged ion. Each ion weighs $107.88/(6.03 \times 10^{23})$ g. (The figure 107.88 is the number of grams of silver in one mole or in 6.03×10^{23} atoms of silver.) Hence the number of Ag ions deposited on the cathode in time T is

$$\text{(No. of) Ag ions} = \frac{M}{\dfrac{107.88}{6.03 \times 10^{23}}} \quad \text{(in time } T\text{)}$$

where M is the increased mass of the cathode in grams.

The charge passing through the circuit in time T is

$$Q = \text{(No. of) Ag ions} \times \frac{\text{charge}}{\text{Ag ion}} = \frac{M \times (6.03 \times 10^{23}) \times (1.6 \times 10^{-19})}{107.88}$$

and

$$I = \frac{Q}{T} = \frac{96{,}480}{107.88} \times \frac{M}{T} \quad \text{amp} \qquad (16)$$

where M is the increased mass of the cathode in grams

T is the time employed in making the test in seconds.

Since both M and T can be measured quite accurately, the ampere value of the scale reading of the pointer of the hot-wire instrument is well defined. A rearrangement of equation (16) will show that a current of 1 amp deposits silver at the rate of 0.001118 g/sec.

If the electrolytic calibration described above is performed in accordance with the specifications contained in Bureau of Standards Bulletin 60, the results can be reproduced with an accuracy of about 1 part in 10,000 in any part of the world. This reliability of calibration is responsible for the following definition:

An international ampere is that value of unvarying current which will deposit silver on the cathode of an $AgNO_3$ cell at the rate of 0.00111800 g/sec.

To the accuracy with which we are here concerned, the international ampere as defined above is identical with the electrostatically derived coulomb per second. (An electrostatically derived coulomb, it will be remembered, is that unit of charge which when placed 1 meter from a similar charge in free space experiences a force of repulsion of 9×10^9 newtons.)

PROBLEMS

1. How many gram-calories of heat are developed in a 50,000-ohm resistor if a current of 100 milliamp flows through the resistor for a period of 10 min? *Note: As a ratio*

$$\frac{\text{(No. of)g-cal}}{\text{watt-sec}} = 0.2388$$

2. How many watt-seconds (or joules) of electrical energy are required to heat 100 lb of water from 42°F to 212°F? $\left(\frac{\text{(No. of)watt-sec}}{\text{Btu}} = 1055\right)$. *Ans.* 17.93×10^6 watt-sec.

3. What is the cost of heating the 100 lb of water in Prob. 2 if electrical energy costs 4 cents per kwhr? Neglect radiation.

4. A small alloy resistor having a negligibly small temperature coefficient of resistivity is immersed in 100 g of water in a calorimeter which is thermally well insulated. A steady current of 1.0 amp flows through the resistor for a period of 10 min. Find the temperature rise of the water if the potential difference across the terminals of the resistor throughout the test is 7.0 volts.

5. Two hundred pounds of copper (initially at 20°C) is to be heated to a temperature of 520°C with an electric heater which is 0.8 efficient. What is the cost of the electrical energy if the charge for this energy is 2 cents per kwhr? (The specific heat of copper is 0.0919, and its melting point is 1083°C.)

6. Instead of raising the temperature of the 200 lb of copper in Prob. 5 to 520°C, it is required to melt the copper. Find the cost of the electrical energy. (The latent heat of fusion of copper is 77.4 Btu/lb.)

7. How many gram-calories of heat are developed at the positive plate of a vacuum tube as a result of 2.5×10^{19} electrons striking the plate if each of these electrons has fallen through a voltage rise of 400 volts? Assume that all the energy possessed by the electrons when they strike the plate is converted into heat by inelastic impacts at the plate.

8. By definition an electron volt of energy is the equivalent of 1 electronic charge falling through a potential difference of 1 volt and is therefore the equivalent of 1.6×10^{-19} joule. A Btu of heat energy is the equivalent of how many electron volts of energy?

9. The ends of an otherwise thermally insulated copper rod (100 cm long and 2 sq cm in cross-section) are maintained at a temperature difference of 50°C. At what rate is heat transferred longitudinally through the rod if the thermal conductivity of the copper is 0.92 g-cal/sec/sq cm/cm/°C. *Note*: This method of designating conductiv-

ity might appear somewhat ambiguous to the beginner, but thermal conductivity is often tabulated in this manner in handbooks. The precise meaning is that c is

$$0.92 \frac{\text{g-cal} \times \text{cm}}{\text{sec} \times \text{sq cm} \times {}^\circ\text{C}}$$

10. If the irregular shaped body shown in Fig. 9 is a piece of aluminum having a thermal conductivity of 2.17 watts/cm³/°C, what is the rate of heat transfer from surface A to surface B in joules per second if these surfaces are maintained at a temperature difference of 40°C? Neglect radiation.

11. A steady current of 40 amp is flowing through 1000 ft of No. 10 B. & S. gage copper wire which is strung out in free air, the temperature of which is 20°C. If the average internal temperature of the copper wire is 60°C, find the centigrade degree rise (above ambient temperature) per watt per square inch.

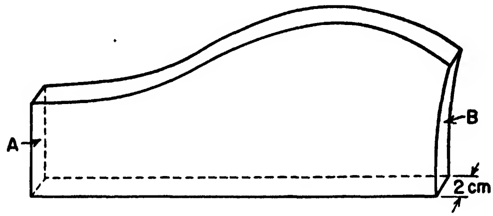


FIG. 9. See Prob. 10.

12. A coil of copper wire which is known to have a resistance of 4 ohms at 20°C has a radiating surface of 25 sq in. With 8 volts potential difference applied to the coil terminals, the current ultimately reaches a steady-state value of 1.5 amp.

(*a*) Find the average steady-state temperature of the copper by the "change of resistance" method.

(*b*) Evaluate $\dfrac{\text{deg C rise}}{\text{watt/sq in.}}$ basing the centigrade degree rise on the average internal temperature of the copper.

13. What is the effective value of the saw-tooth current variation shown in Fig. 10. [*Note*: $i = (I_m/T)t$.] Express result in terms of I_m.

14. If, in Fig. 10, I_m is numerically equal to 3 amp and this current is flowing through a 10-ohm resistor, what number of joules of energy is converted into heat in a 10-min period? Would a steady or direct current of 1.5 amp produce more heat, less heat, or the same amount of heat in the resistor in the same period of time?

15. What is the effective value of the current variation shown in Fig. 11 expressed in terms of I_m?

16. What is the effective value of the current variation shown in Fig. 12?

17. Given a 100-volt generator and a rotating contactor *something* like that shown in Fig. 5, page 102, draw the circuit arrangement which can be used to produce the current-time graph shown in Fig. 12. Specify the rotation of the contactor in rpm and the ohmic values of the three resistors that are employed to obtain the desired variation of I.

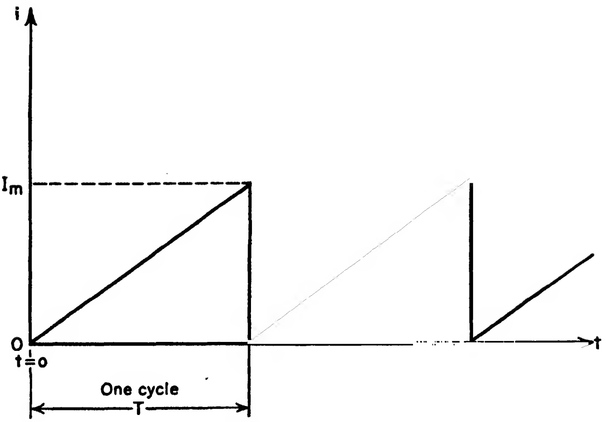


FIG. 10. See Probs. 13, 14, and 20.

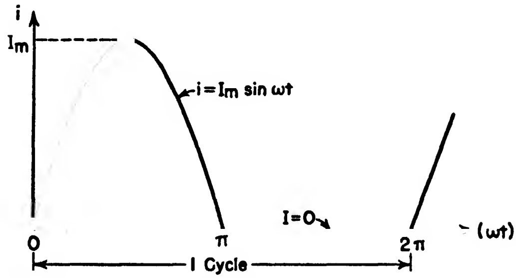


FIG. 11. See Probs. 15 and 19.

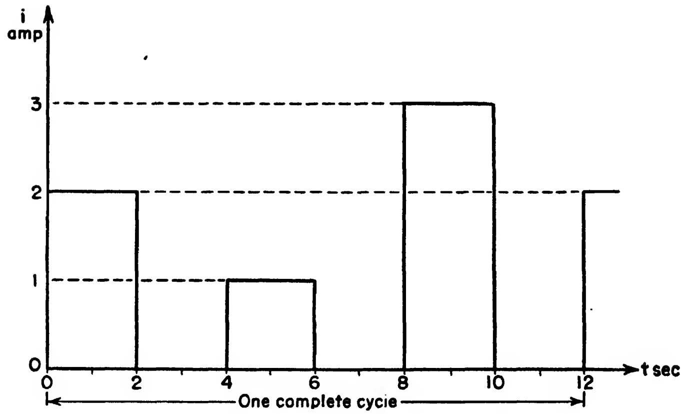


FIG. 12. See Probs. 16, 17, and 18.

18. What is the time-averaged value of the current variation shown in Fig. 12 taken over a complete cycle?

19. What is the time-averaged value of the current variation shown in Fig. 11 taken over a complete cycle if $I_m = \pi$ amp?

20. What is the time-averaged value of the current shown in Fig. 10 taken over a period of 2 complete cycles if $I_m = 3$ amp?

21. Throughout a 40-min period, a current

$$i = 10 + 8 \sin 377t \text{ amp}$$

flows through a hot-wire ammeter and a coulombmeter in series as shown in Fig. 8, page 108.

(a) What is the reading of the hot-wire ammeter, if it is assumed that the instrument has been properly calibrated?

(b) What number of grams of silver is deposited on the cathode of the coulombmeter during the 40-min period?

(c) What would be the reading of a permanent-magnet type of ammeter (one which indicates average current) if it were placed in the series circuit?

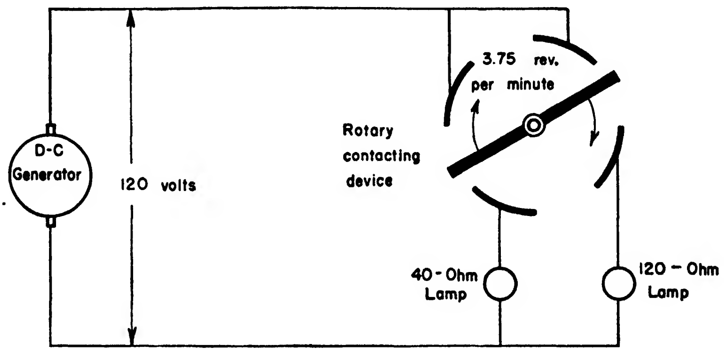


FIG. 13. See Prob. 22.

22. *Form factor* of a time-varying current or voltage is *defined* as

$$\frac{\text{effective value}}{\text{average value}}$$

What is the *form factor* of the current which flows through the d-c generator shown in Fig. 13 if the average value in the above definition is taken as the average value of current over one complete cycle?

23. Refer to the definition of *form factor* given in Prob. 22. What is the *form factor* of the current-time graph shown in Fig. 14 if the average value referred to in the definition is taken as the half-cycle average? What answer would be obtained in this case if the average value were taken as the full-cycle average?

24. (a) What are the values of resistors R_1, R_3, and R_4 in Fig. 15 if R_2 is known to be 3000 ohms?

(*b*) What is the total power delivered to the system by the battery E. (This battery has a potential difference of 410 volts across its terminals to maintain the potential distribution shown in Fig. 15.)

(*c*) If 1-, 5-, and 10-watt, 10,000-ohm resistors are available, which would you use for R_3?

25. In addition to the data specified in Fig. 15, it will be assumed that R_3 is known to be 5000 ohms. With both the 100-ma and 10-ma loads connected as shown, the potential distribution is as given in Fig. 15.

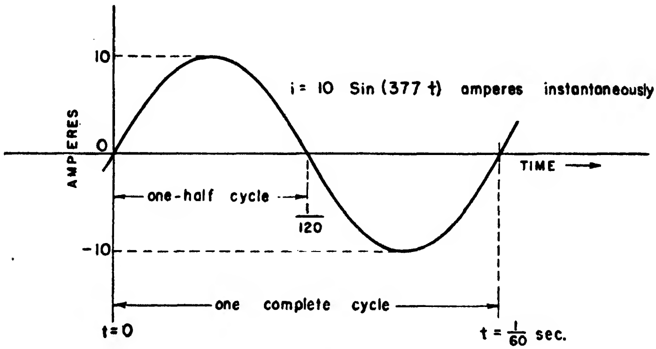


FIG. 14. See Prob. 23.

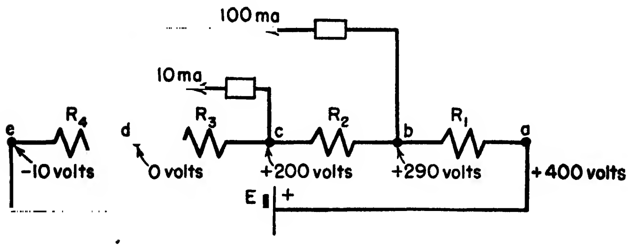


FIG. 15. See Probs. 24 and 25.

What would be the potentials of points *a*, *b*, and *e* relative to point *d* if the 10-ma path were to be accidentally opened? (It will be assumed that E remains fixed at 410 volts and that the 100-ma load connected between points *b* and *d* is unaffected by the potential redistribution which occurs when the 10-ma load is reduced to zero by the open circuit.)

CHAPTER VI

D-C Circuit Analysis (Linear-Bilateral Circuit Elements)

1. Classification of Circuit Elements. In general, a linear system is one in which effects are directly proportional to causes, that is, linearly related to causes. If the current that passes through a circuit element is directly proportional to the potential difference applied to its terminals, the element is said to be *linear*; otherwise it is said to be *non-linear*.

A circuit element is *bilateral* or *unilateral* depending upon whether it passes current equally well in both directions or in one direction only.

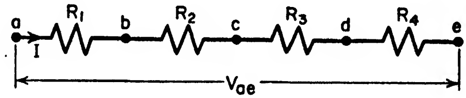


FIG. 1. Four resistances in series.

Circuit elements which transmit equally well in both directions are said to be *bilateral*. Circuit elements like vacuum tubes and rectifiers which transmit effectively in one direction only are said to be *unilateral*. [Vacuum tubes may be so operated as to function as *linear unilateral* elements (to alternating current) as will be shown in Chapter XV.]

Non-linear and unilateral circuit elements will be considered to some extent in later chapters. The present chapter is confined to systematized solutions of *linear-bilateral* network problems.

2. Series Circuits. Where several resistors are connected in series (that is, carry the same current), it is usually convenient in circuit analysis to replace these resistances with an equivalent single resistance. As applied to the terminals a and e of Fig. 1, it is plain that

$$V_{ab} + V_{bc} + V_{cd} + V_{de} = V_{ae} \tag{1}$$

or

$$R_1I + R_2I + R_3I + R_4I = V_{ae} \tag{2}$$

The equivalent resistance between terminals a and e of Fig. 1 is

$$R_{\text{eq. } a-e} = R_1 + R_2 + R_3 + R_4 = \frac{V_{ae}}{I} \tag{3}$$

and in general

$$R_{\text{eq. (series)}} = R_1 + R_2 + R_3 + R_4 + R_5 + \cdots \tag{4}$$

Equation (4) indicates the rather obvious fact that any number of resistances which are joined in series may be treated as a single equivalent resistance for the purposes of analysis.

3. Equivalent Resistance. The *equivalent resistance* between any two terminals of a passive[1] electrical network is defined as the ratio of the unvarying voltage between these two terminals to the current flowing into (or out of) one of these terminals. The current flowing into one terminal will of course return at the other terminal if these two terminals are energized with, say, a two-terminal battery or generator.

A simple example of this definition of equivalent resistance has been given in connection with Fig. 1. [See equation (3).] As defined above *equivalent resistance* may refer to any two terminals of a network regardless of the configuration of the network.

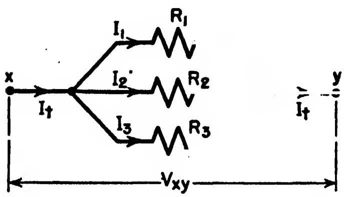


FIG. 2. Three resistances in parallel.

The equivalent resistance between two terminals is sometimes referred to as the *d-c driving point resistance*. The qualification to *d-c resistance* is necessary here because condensers which might very well be used in a-c circuits would block the direct current and the a-c resistance between the two terminals would then be distinctly different from the d-c resistance.

4. Parallel Circuits. Each of the three resistors shown in Fig. 2 is subjected to the same potential difference, V_{xy}. Therefore

$$I_1 = \frac{V_{xy}}{R_1} \qquad I_2 = \frac{V_{xy}}{R_2} \qquad I_3 = \frac{V_{xy}}{R_3} \tag{5}$$

If Kirchhoff's current law is applied to one of the two junctions shown in Fig. 2, it is plain that

$$I_t = I_1 + I_2 + I_3 = \left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}\right) V_{xy} \tag{6}$$

[1] A *passive* network is one which is devoid of active generators or other sources of emf.

The ratio of V_{xy} to I_t is

$$R_{\text{eq.}x-y} = \frac{V_{xy}}{I_t} = \frac{1}{\dfrac{1}{R_1} + \dfrac{1}{R_2} + \dfrac{1}{R_3}} = \frac{R_1R_2R_3}{R_1R_2 + R_2R_3 + R_3R_1} \tag{7}$$

In general, the equivalent resistance of several branches in parallel is

$$R_{\text{eq.(par.)}} = \frac{1}{\dfrac{1}{R_1} + \dfrac{1}{R_2} + \dfrac{1}{R_3} + \dfrac{1}{R_4} + \dfrac{1}{R_5} + \cdots} \tag{8}$$

If the number of parallel branches is limited to two,

$$R_{\text{eq.}} = \frac{R_1R_2}{R_1 + R_2} \tag{9}$$

One point to observe in connection with resistances in parallel is that the equivalent resistance is always less than the smallest of the individual resistances. With the aid of the above equations, any number of parallel branches between two terminals can be reduced quickly to an equivalent single resistance between these two terminals.

Example. If the three resistances shown in Fig. 2 are respectively 2, 4, and 10 ohms, they may be replaced with an equivalent resistance:

$$R_{\text{eq.}} = \frac{2 \times 4 \times 10}{(2 \times 4) + (4 \times 10) + (10 \times 2)} = \frac{80}{68} = 1.17\dot{7} \quad \text{ohms}$$

5. Conductance. As pointed out on page 78, the conductivity of a material (γ) is the reciprocal of the resistivity (ρ). Whereas the d-c resistance of a conductor is

$$R = \rho \frac{l}{A}$$

the d-c *conductance* of the conductor is

$$G = \gamma \frac{A}{l} \tag{10}$$

where A is the cross-sectional area of the conductor
l is the length of the conductor
γ is the conductivity of the material.

The primary unit of conductance in the mks system of units is the *mho*, and physically it represents the number of amperes that will flow

through a conductor per volt of d-c potential difference applied to the terminals of that conductor. Since $G = 1/R$, it is plain that in those cases where unvarying currents and voltages are involved

$$I = GV \tag{11}$$

In some types of circuit analysis, it is more convenient to use conductance than it is to use resistance. Where several resistances are in parallel, the use of conductance may simplify the calculations to some extent.

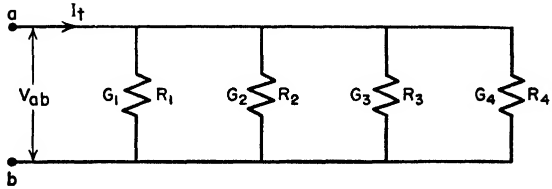


FIG. 3. $G_{eq.ab} = G_1 + G_2 + G_3 + G_4$ and $I_t = G_{eq.}V_{ab}$.

Example. Four resistances are connected in parallel across a pair of terminals ($a \rightarrow b$) as shown in Fig. 3. Let it be required to find I_t if $R_1 = 1$, $R_2 = 0.5$, $R_3 = 2$, $R_4 = 4$ ohms, and $V_{ab} = 20$ volts.

$$G_{eq.} = G_1 + G_2 + G_3 + G_4 = 1 + 2 + 0.5 + 0.25 = 3.75 \quad \text{mhos}$$

$$I_t = G_{eq} V_{ab} = 3.75 \times 20 = 75 \quad \text{amp}$$

6. Internal Generator Resistance. Thus far no attention has been paid to the internal resistance of the driving source. The electrolyte and plates of a battery, for example, present an impedance to the flow of

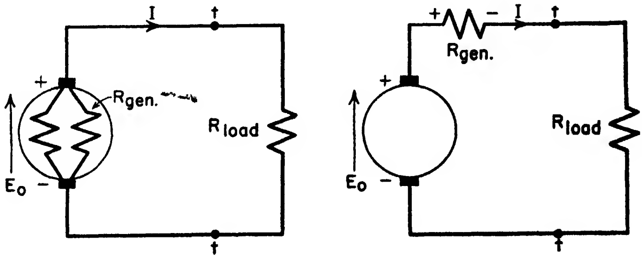


FIG. 4. Replacement of internal resistance with a series external resistance.

current. The battery, therefore, has an internal resistance which must be taken into account in certain problems, particularly in those cases where only the open-circuit voltage of the battery is specified.

Similarly, the resistance of the armature windings of a d-c generator (symbolized by the two parallel resistive paths in Fig. 4) must be taken

into account if the open-circuit generated voltage only is specified. The internal resistance of a generator (or battery) can be accounted for quite easily by noting that a current flowing through the internal resistance establishes a voltage drop which subtracts directly from the open-circuit voltage to give the terminal voltage. As shown in Fig. 4, the internal resistance can be placed in series with the open-circuit voltage E_0 and the terminal voltage of the generator under load conditions calculated as

$$E_{tt} = E_0 - R_{\text{gen}} I \tag{12}$$

Thus the open-circuit (or no-load) voltage of a generator may be treated as the voltage of an ideal generator provided R_{gen} is placed directly in series with E_0 as shown in Fig. 4.

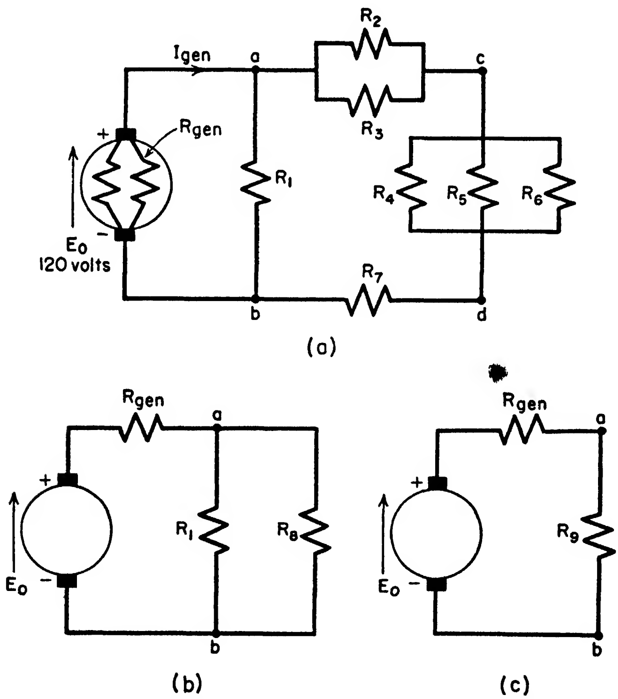


FIG. 5. Reduction of a network to a single equivalent resistance.

7. Reduction of Series-Parallel Resistances to a Single $R_{\text{eq.}}$. Many networks can be reduced to a single equivalent resistance in series with an open-circuit generated voltage by the simple principles contained in Sections 2, 4, and 6. An example of this kind of reduction is shown in

Fig. 5. The details are presented below for the case of

$$R_1 = 5 \text{ ohms} \quad R_4 = 2 \text{ ohms} \quad R_7 = 2.6(23) \text{ ohms}$$
$$R_2 = 2 \text{ ohms} \quad R_5 = 4 \text{ ohms} \quad R_{\text{gen.}} = 0.5 \text{ ohm}$$
$$R_3 = 3 \text{ ohms} \quad R_6 = 10 \text{ ohms} \quad E_0 = 120 \text{ volts}$$

$$R_{acdb} = R_8 = R_{ac} + R_{cd} + R_{db}$$
$$\frac{2 \times 3}{2+3} + \frac{2 \times 4 \times 10}{8 + 40 + 20} + 2.623$$
$$= 1.2 + 1.177 + 2.623 = 5.0 \text{ ohms}$$

$$R_{ab} = R_9 = \frac{R_1 R_8}{R_1 + R_8} = \frac{5 \times 5}{5 + 5} = 2.5 \text{ ohms}$$

$$I_{\text{gen}} = \frac{E_0}{R_{\text{gen.}} + R_9} = \frac{120}{0.5 + 2.5} = 40 \text{ amp}$$

To find the various branch currents, the circuit is re-established in the reverse order as

$$V_{ab} = E_{tt} = I_{\text{gen.}} R_9 = 40 \times 2.5 = 100 \text{ volts}$$
$$I_1 \text{ (current through } R_1) = 100/5 = 20 \text{ amp}$$
$$V_{ac} \text{ (voltage across } R_2, R_3) = 20 \times 1.2 = 24 \text{ volts}$$
$$I_2 \text{ (current through } R_2) = 24/2 = 12 \text{ amp}$$
$$I_3 \text{ (current through } R_3) = 24/3 = 8 \text{ amp}$$

From Fig. 5–a, it is plain that $I_{\text{gen.}} = I_1 + I_2 + I_3$; as the above arithmetic shows.

Exercise. Find the currents in R_4, R_5, R_6, and R_7 of Fig. 5-*a* and check the result by Kirchhoff's current law.

8. Network Reduction by the Delta-Wye Transformation. In many instances, the simple type of network reduction shown in Section 7 cannot be employed because of cross connections which make it impossible to effect a simple series-parallel type of reduction. In these cases the reduction can sometimes be carried out by means of a delta-to-wye transformation.

A delta-to-wye transformation involves the replacement of a three-sided mesh (or loop) like $R_1 - R_2 - R_3$ of Fig. 6 with a three-legged wye arrangement like $R_x - R_y - R_z$ which connects to the same three terminals of the network as does the mesh which is being replaced. A

three-sided mesh, or delta as it is usually called, like $R_1 - R_2 - R_3$ of Fig. 6 may be *replaced* by a three-legged (or wye) configuration like $R_x - R_y - R_z$ provided the values of R_x, R_y, and R_z are properly chosen. A substitution of this kind demands that the equivalent resistances looking into terminals xy, yz, and zx be precisely the same after the substitution is made as they were originally. The substitution is valid if

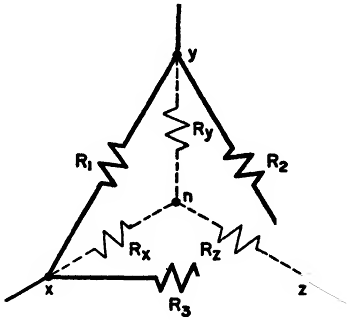


FIG. 6. $R_1 - R_2 - R_3$ are delta-connected resistances which are to be replaced with wye-connected resistances $R_x - R_y - R_z$.

(looking into terminals xy)

$$R_x + R_y = \frac{R_1(R_2 + R_3)}{R_1 + R_2 + R_3} \qquad (13)$$

(looking into terminals yz)

$$R_y + R_z = \frac{R_2(R_1 + R_3)}{R_1 + R_2 + R_3} \qquad (14)$$

(looking into terminals zx)

$$R_z + R_x = \frac{R_3(R_1 + R_2)}{R_1 + R_2 + R_3} \qquad (15)$$

Equations (13), (14), and (15) may be solved simultaneously for R_x, R_y, and R_z in terms of the original delta resistances R_1, R_2, and R_3. If, for example, equation (14) is subtracted from equation (15), there is obtained

$$R_x - R_y = \frac{R_1R_3 + R_2R_3 - R_1R_2 - R_2R_3}{R_1 + R_2 + R_3} \qquad (16)$$

which, if added to equation (13), results in

$$R_x = \frac{R_1R_3}{R_1 + R_2 + R_3} \qquad (17)$$

and

$$R_y = \frac{R_1R_2}{R_1 + R_2 + R_3} \qquad (18)$$

$$R_z = \frac{R_2R_3}{R_1 + R_2 + R_3} \qquad (19)$$

Equations (17), (18), and (19) may be used to effect a delta-to-wye transformation. In making this type of transformation, the equivalent wye resistance which connects to a given terminal is equal to the product of the two delta resistances which connect to this same terminal divided

by the sum of the delta resistances. It is much easier to learn this rule than to adapt equations (17), (18), and (19) to different subscripts which are encountered in different networks.

Example. The bridge circuit shown in Fig. 7-a cannot be reduced to an equivalent series circuit between the terminals a and d by the method pre-

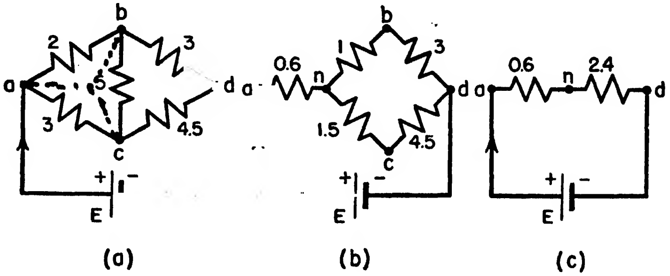


FIG. 7. Reduction of Wheatstone bridge circuit. (Numbers indicate ohms.)

sented in Section 7 because of the cross-connection from b to c. The reduction can, however, be made easily with the aid of a delta-to-wye transformation as shown below.

Either the abc delta or the bcd delta may be reduced to an equivalent wye. In this example the abc delta will be reduced. Simply by inspection of the abc delta, it is possible to write

$$R_{an}(\text{of Fig. 7-b}) = \frac{2 \times 3}{2 + 3 + 5} = 0.6 \quad \text{ohm}$$

$$R_{bn}(\text{of Fig. 7-b}) = \frac{2 \times 5}{10} = 1.0 \quad \text{ohm}$$

$$R_{cn}(\text{of Fig. 7-b}) = \frac{5 \times 3}{10} = 1.5 \quad \text{ohms}$$

$$R_{nd}(\text{of Fig. 7-c}) = \frac{4 \times 6}{4 + 6} = 2.4 \quad \text{ohms}$$

$$R_{ad}(\text{of Fig. 7-c}) = 0.6 + 2.4 = 3.0 \quad \text{ohms}$$

Since R_{ad} of Fig. 7-c is equivalent to $R_{eq.ad}$ of Fig. 7-a, the battery current in Fig. 7-a may be found to be $E/3$ amp, just as if a single resistance of 3 ohms were connected across the battery terminals. After the battery current has been evaluated, V_{bc} can be quickly found and the circuit reassembled to find any or all of the branch currents.

9. Balanced Wheatstone Bridge. The circuit arrangement shown in Fig. 7-a is known as a Wheatstone bridge and is widely used for resistance measurements by the comparison method. A sensitive galvanom-

eter is placed in the bc branch to indicate by a null reading when zero potential difference exists between points b and c of the network. If points b and c are at the same potential, it follows that $I_{bc} = 0$, $I_{bd} = I_{ab}$, and $I_{cd} = I_{ac}$. Hence

$$R_{ab}I_{ab} = R_{ac}I_{ac} \tag{20}$$

and

$$R_{bd}I_{ab} = R_{cd}I_{ac} \tag{21}$$

Dividing equation (20) by equation (21) gives

$$\frac{R_{ab}}{R_{bd}} = \frac{R_{ac}}{R_{cd}} \tag{22}$$

If now R_{ab}/R_{bd} is some fixed ratio, R_{ac} can be measured in terms of an adjustable calibrated resistance, R_{cd}, as

$$R_{ac} = \frac{R_{ab}}{R_{bd}} R_{cd} \tag{23}$$

In practice one frequently encounters balanced bridge circuits where it is necessary to have the potential difference between one pair of terminals like bc unaffected by a change in voltage between another pair of terminals like ad of Fig. 7-a.

10. Network Solutions by the Branch-Current Method. The direct application of Kirchhoff's laws to networks provides a general method of solution. If the actual branch currents are selected as the dependent or unknown quantities, the *number of unknowns* equals the *number of branches*, and the problem is that of finding an explicit expression for each branch current in terms of the specified voltages and resistances. In this connection a branch is a conducting path between two junctions of the network, and a branch current is the current in one of the branches. It is the current that an ammeter would read if placed in the branch.

The required number of independent equations (necessary to effect a solution) can be obtained straightforwardly if an orderly sequence of steps is taken. The network shown in Fig. 8 will be employed in explaining this sequence of steps.

(1) Simplify the network as far as possible by replacing all series and parallel combinations of resistances that exist between junctions with equivalent series-circuit resistances between the junctions. As applied to Fig. 8 this means simply that we treat the resistance of the *fabc* branch as 13 ohms and the resistance of the *cdef* branch as 10 ohms. (The internal resistances of the generators in Fig. 8 are considered to be negligibly small.)

(2) Decide, from an inspection of the network diagram, the exact number of branches, that is, the number of paths between junctions. For the case shown in Fig. 8 there are *three* branches, namely, *fabc*, *cdef*, and *cf* after the reduction suggested in (1) has been made.

(3) Label each branch current as shown in Fig. 8 and *arbitrarily* assign an arrow direction to each branch current. Where the correct direction of $+I$ is clearly evident from the polarities of the emf sources as in branches 1 and 2 of Fig. 8, it is convenient but not necessary to employ this direction as the arrow direction of the current. I_3 in Fig. 8, for example, has purposely been assigned an arrow direction from c to f even though the actual positive current direction in this branch will turn out to be from f to c. An *assigned* direction of current is not necessarily the positive current direction, as will be shown presently.

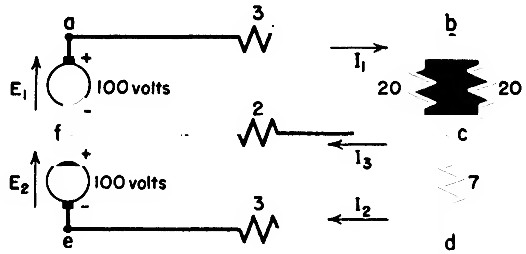


FIG. 8. Illustrating branch currents I_1, I_2, and I_3.

(4) Obtain simple relationships between the branch currents by applying Kirchhoff's current law *one less times than the network has junctions,* thus obtaining as many independent current relations as possible from the current law. As applied to Fig. 8, only one application of the current law is made since Fig. 8 is a two-junction network.

At junction c:
$$I_1 = I_2 + I_3 \tag{24}$$

(5) Apply Kirchhoff's voltage law to the loops of the network as many times as the network has branches less the number of times the current law has been used. (See step 4.) If the network has N branches and J junctions, the current law is used $J - 1$ times and the voltage law $N - (J - 1)$ times. In so doing the N required independent relationships between the branch currents are established.

In writing the voltage law, one *new* branch must be incorporated in each loop equation in order to obtain *independent* loop or voltage equations. As applied to Fig. 8, two loop equations must be written.

For loop *abcfa*:
$$R_1I_1 + R_3I_3 = E_1 \tag{25}$$

For loop *fcdef*:
$$-R_3I_3 + R_2I_2 = E_2 \tag{26}$$

A third loop equation might be written (around *abcdefa*) but it will not be independent of equations (25) and (26) because no branch not contained in equations (25) and (26) would be encountered.[2]

(6) Solve the simultaneous equations (N in number) for the branch currents, $I_1, I_2, I_3, \cdots, I_N$.

As applied to Fig. 8, equations (24), (25), and (26) may be solved simultaneously for I_1, I_2, and I_3. In a case as simple as the present one, a systematic elimination can be employed to obtain the values of I_1, I_2, and I_3. Thus, I_3 in equations (25) and (26) may be replaced with $I_1 - I_2$, the value of I_3 obtained from equation (24). Then

$$R_1I_1 + R_3(I_1 - I_2) = E_1 \tag{27}$$

$$-R_3(I_1 - I_2) + R_2I_2 = E_2 \tag{28}$$

or

$$(R_1 + R_3)I_1 - R_3I_2 = E_1 \tag{27-a}$$

$$-R_3I_1 + (R_2 + R_3)I_2 = E_2 \tag{28-a}$$

Substitution of numerical values in equations (27-a) and (28-a) gives

$$15I_1 - 2I_2 = 100 \tag{27-b}$$

$$-2I_1 + 12I_2 = 100 \tag{28-b}$$

From which

$$I_1 = 7.95 \text{ amp} \quad \text{and} \quad I_2 = 9.66 \text{ amp}$$

$$I_3 = I_1 - I_2 = 7.95 - 9.66 = -1.71 \text{ amp}$$

The minus sign in connection with I_3 means simply that the arrow direction of I_3 (which was *arbitrarily* assigned initially) is opposite to the actual positive direction of I_3. The actual positive direction of current flow in branch 3 of Fig. 8 is from f to c.

The branch-current method of solution (as outlined above) is very useful in many types of network problems. It is easy to apply generally, and after a little practice it becomes a routine procedure. If care is not taken in writing the voltage equations, however, some trouble is likely to be encountered with *signs*. (See page 58.)

[2] In writing the loop equations, either direction around a loop may be taken as the *tracing* direction. Normally, fewer minus signs will be encountered if the tracing direction is taken in the − to + direction through the generator. An RI voltage written on the left-hand side of the loop equation like equation (26) is + if the assigned arrow direction of the branch current and the tracing direction coincide; otherwise this RI drop is negative. An emf written on the right-hand side of the loop equation is + if the tracing direction is in the − to + direction through this emf; otherwise this emf is written into the equation as a negative quantity.

A more highly systematized method of solution is given in the following section.

11. The Mesh-Current Method of Solution. If, instead of branch currents, *loop* or *mesh* currents are employed, the necessity for applying the current law is eliminated. This elimination is particularly advantageous in networks having several junctions. Network solutions in terms of *mesh* currents are generally applicable and so widely used in practice that every student of electrical engineering should master the details of this type of solution.

A loop or mesh current is a continuous current around any closed loop of the network. The algebraic sum of the mesh currents flowing in any branch is the actual branch current, that is,

$$\Sigma I_{\text{mesh}} = I_{\text{branch}} \tag{29}$$

The significance of mesh currents and the reason why they can be used in place of branch currents are illustrated in Fig. 9 where I_1 and I_2 are the

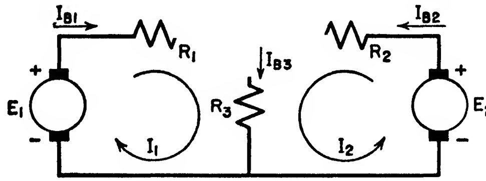


FIG. 9. Illustrating loop or mesh currents I_1 and I_2 and branch currents $I_{B1} = I_1$, $I_{B2} = I_2$, and $I_{B3} = I_1 + I_2$.

mesh currents and I_{B1}, I_{B2}, I_{B3} are the actual branch currents. If Kirchhoff's voltage law is applied to loops 1 and 2 of Fig. 9 by use of the branch currents shown, the following equations are obtained:

$$R_1 I_{B1} + R_3 I_{B3} = E_1 \tag{30}$$

$$R_2 I_{B2} + R_3 I_{B3} = E_2 \tag{31}$$

In any branch, the actual branch current is the algebraic sum of the mesh currents flowing through that branch so equations (30) and (31) may be written in terms of mesh currents as

$$R_1 I_1 + R_3(I_1 + I_2) = E_1 \quad \text{or} \quad (R_1 + R_3)I_1 + R_3 I_2 = E_1 \tag{32}$$

$$R_2 I_2 + R_3(I_1 + I_2) = E_2 \quad \text{or} \quad R_3 I_1 + (R_2 + R_3)I_2 = E_2 \tag{33}$$

From equations (32) and (33), one observes that Kirchhoff's laws can be written just as well in terms of mesh currents (I_1 and I_2) as in terms of the branch currents (I_{B1}, I_{B2}, and I_{B3}). The requirement is that the entire resistance of loop 1, $(R_1 + R_3)$ here, be multiplied by loop current 1; and that the portion of the resistance of loop 1 through

which loop current 2 (I_2) flows be multiplied by I_2 in forming equation (32). In this manner, the voltage law is satisfied and the necessity for employing the current law is eliminated.

The great advantage of the loop-current method of solution is that, after a little practice with it, the determinant form of solution for any mesh or loop current can be written down from an inspection of the circuit arrangement. The method is so systematized that the need for writing down the voltage equations is even eliminated.

The voltage equations which apply to a three-loop network are shown in equations (34). From the system with which the subscripts are written, it should be evident how to contract these equations to a two-loop network and how to expand them to include a four- or five-loop network.

$$\begin{aligned} R_{11}I_1 + R_{12}I_2 + R_{13}I_3 &= E_1 \\ R_{21}I_1 + R_{22}I_2 + R_{23}I_3 &= E_2 \\ R_{31}I_1 + R_{32}I_2 + R_{33}I_3 &= E_3 \end{aligned} \tag{34}$$

where I_1, I_2, and I_3 are the arbitrarily assigned loop currents

R_{11} is the total resistance of loop 1 to current I_1

R_{22} is the total resistance of loop 2 to current I_2

R_{33} is the total resistance of loop 3 to current I_3

$R_{12} = R_{21}$ is the resistance which is common to loops 1 and 2

$R_{13} = R_{31}$ is the resistance which is common to loops 1 and 3

$R_{23} = R_{32}$ is the resistance which is common to loops 2 and 3

E_1, E_2, and E_3 are the driving voltages of loops 1, 2, and 3 respectively, considered positive on the right-hand side of equations (34) if traversed in the $-$ to $+$ direction when the loop is traced in the arrow directions which have been assigned to the loop currents.

For any particular network, the number of equations, like those shown in (34) required to effect a solution is equal to the number of partially independent loops in the network.

After the required number of equations have been arranged as shown in (34), the solutions for I_1, I_2, and I_3 may be effected straightforwardly as shown below:

$$I_1 = \frac{\begin{vmatrix} E_1 & R_{12} & R_{13} \\ E_2 & R_{22} & R_{23} \\ E_3 & R_{32} & R_{33} \end{vmatrix}}{\begin{vmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{vmatrix}} = \frac{\begin{vmatrix} E_1 & R_{12} & R_{13} \\ E_2 & R_{22} & R_{23} \\ E_3 & R_{32} & R_{33} \end{vmatrix}}{D} \tag{34-a}$$

where

$$D = \begin{vmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{vmatrix} = \begin{array}{l} R_{11}R_{22}R_{33} + R_{12}R_{23}R_{31} + R_{13}R_{32}R_{21} \\ - R_{13}R_{22}R_{31} - R_{12}R_{21}R_{33} - R_{11}R_{32}R_{23} \end{array}$$

and

$$\begin{vmatrix} E_1 & R_{12} & R_{13} \\ E_2 & R_{22} & R_{23} \\ E_3 & R_{32} & R_{33} \end{vmatrix} = \begin{array}{l} E_1R_{22}R_{33} + R_{12}R_{23}E_3 + R_{13}R_{32}E_2 \\ - R_{13}R_{22}E_3 - R_{12}E_2R_{33} - E_1R_{32}R_{23} \end{array}$$

Similarly

$$I_2 = \frac{\begin{vmatrix} R_{11} & E_1 & R_{13} \\ R_{21} & E_2 & R_{23} \\ R_{31} & E_3 & R_{33} \end{vmatrix}}{D} \quad \text{and} \quad I_3 = \frac{\begin{vmatrix} R_{11} & R_{12} & E_1 \\ R_{21} & R_{22} & E_2 \\ R_{31} & R_{32} & E_3 \end{vmatrix}}{D} \qquad (34\text{–b})$$

It will be observed that the denominator matrix D is common to all of the loop current solutions and that D can be evaluated solely in terms of

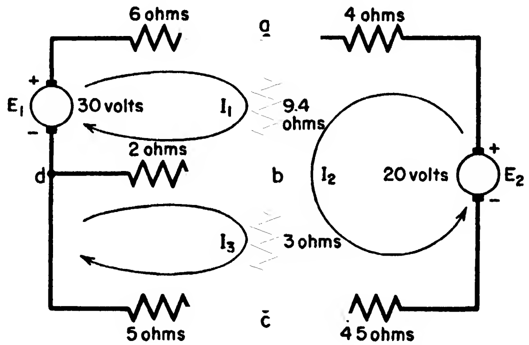


FIG. 10. Illustrating loop currents I_1, I_2, and I_3.

the R's. The numerator of I_1 is obtained by writing D with the *first column* replaced by the E's. The numerator of the I_2 solution is obtained by writing D with the *second column* replaced by the E's which appear on the right-hand sides of equations (34), and so on.[3]

Example. As applied to Fig. 10, the specific values to employ in a mesh- or loop-current solution are

$$R_{11} = 6 + 9.4 + 2 = 17.4 \quad \text{ohms}$$

$$R_{22} = 4 + 9.4 + 3 + 4.5 = 20.9 \quad \text{ohms}$$

[3] For further details regarding the techniques employed in connection with determinants, the reader is referred to standard algebra texts or to *Handbook of Engineering Fundamentals*, pp. 2–16, 2–17, by Eshbach, published by John Wiley and Sons.

$$R_{33} \doteq 2 + 3 + 5 = 10 \quad \text{ohms}$$
$$R_{12} = R_{21} = 9.4 \quad \text{ohms}$$
$$R_{13} = R_{31} = 2 \quad \text{ohms}$$
$$R_{23} = R_{32} = 3 \quad \text{ohms}$$
$$E_1 = 30, \quad E_2 = 20, \quad \text{and} \quad E_3 = 0 \quad \text{volts}$$

Hence,

$$17.4I_1 + 9.4I_2 - 2I_3 = 30$$
$$9.4I_1 + 20.9I_2 + 3I_3 = 20$$
$$-2I_1 + 3I_2 + 10I_3 = 0$$

The RI terms in the above equations are positive if the loop current which establishes the RI drop in question is directed along the positive tracing direction which is used in writing the loop equation. The positive tracing directions in the present instance are taken in the arrow directions of the loop currents, I_1, I_2, and I_3 in order that the E's which appear on the right-hand sides of the equations will carry positive signs. In tracing clockwise around loop 1, for example,

$R_{11}I_1$ is positive since I_1 is directed clockwise—positive tracing direction in this case

$R_{12}I_2$ is positive since I_2 is directed from a to b (clockwise) in that portion of loop 1 through which I_2 flows, namely R_{ab}

$R_{13}I_3$ is negative since I_3 is directed from d to b (counterclockwise) in that portion of loop 1 through which I_3 flows, namely R_{db}

The loop currents I_1, I_2, and I_3 shown in Fig. 10 may now be determined straightforwardly as shown below. (Although the accuracy with which the results are stated is not consistent with the accuracy with which the original data are specified, the four-significant-figure results are useful forms in making certain numerical checks later on.)

$$I_1 = \frac{\begin{vmatrix} 30 & 9.4 & -2 \\ 20 & 20.9 & 3 \\ 0 & 3 & 10 \end{vmatrix}}{\begin{vmatrix} 17.4 & 9.4 & -2 \\ 9.4 & 20.9 & 3 \\ -2 & 3 & 10 \end{vmatrix}} = \frac{4000}{2400} = 1.667 \text{ amp}$$

$$I_2 = \frac{\begin{vmatrix} 17.4 & 30 & -2 \\ 9.4 & 20 & 3 \\ -2 & 0 & 10 \end{vmatrix}}{2400} = \frac{400}{2400} = 0.1667 \text{ amp}$$

$$I_3 = \frac{\begin{vmatrix} 17.4 & 9.4 & 30 \\ 9.4 & 20.9 & 20 \\ -2 & 3 & 0 \end{vmatrix}}{2400} = \frac{680}{2400} = 0.2835 \text{ amp}$$

The branch currents follow directly from the solutions given above.

Current through E_1 branch $= I_1 = 1.667$ amp
Current through branch $ab = I_1 + I_2 = 1.667 + 0.1667 = 1.833$ amp
Current through branch $bd = I_1 - I_3 = 1.667 - 0.2835 = 1.383$ amp
Current through branch $bc = I_2 + I_3 = 0.1667 + 0.2835 = 0.450$ amp
Current through branch $cd = I_3 = 0.2835$ amp
Current through E_2 branch $= I_2 = 0.1667$ amp

Exercise. Check the arithmetical accuracy of the branch currents I_{da}, I_{ab}, and I_{db} in the above example by comparing $6I_{da} + 9.4I_{ab} + 2I_{bd}$ with the specified value of E_1.

The correct number of independent loop-current equations required in any particular network is equal to the number of emf equations required after the current law has been applied at the junctions as many times as it can be applied and still obtain independent relationships between the branch currents. As shown on page 124, this latter number is $J - 1$, where J is the number of junctions in the network. If B is the number of branches, the correct number of loop-current equations to employ is

$$B - (J - 1) = B - J + 1 \tag{35}$$

In Fig. 10, for example, $B = 6$, $J = 4$, and the required number of loop-current equations is 3.

Independent loop-current equations are obtained by successively choosing loops that always include at least one branch which has not previously been traced by a mesh or loop current. In following this procedure, it is possible to include too many branches with a single loop current to obtain physically correct results, but when this is done the correct number of loop-current equations $(B - J + 1)$ will not have been established. (See Prob. 38 at the close of the chapter.)

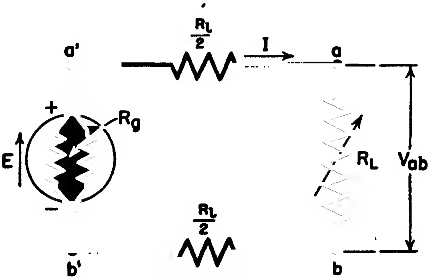


FIG. 11. A two-wire transmission system.

12. Two-Wire Transmission of Electrical Power. Certain aspects of two-wire transmission will be considered in this section. For purposes of illustration the generator shown in Fig. 11 will be assumed to generate an open-circuit voltage of E volts and have an internal resistance of R_g ohms. The generator is connected to a load resistance R_L through a line resistance R_l. Both R_g and R_l are assumed to be fixed, whereas the load resistance R_L is assumed to be variable.

It is plain that

$$I = \frac{E}{R_{gl} + R_L} \tag{36}$$

in which $R_{gl} = R_g + R_l$. The terminal voltage of the generator for any specified load current I is

$$V_{a'b'} = E - R_g I \tag{37}$$

The voltage across the load resistance is

$$V_{ab} = E - R_{gl} I = R_L I \tag{38}$$

The total generated power is

$$P_{\text{gen.}} = EI \tag{39}$$

The power delivered to terminals $a'b'$ is

$$P_{a'b'} = V_{a'b'} I = EI - R_g I^2 \tag{40}$$

The power delivered to the load is

$$P_{ab} = V_{ab} I = EI - R_g I^2 - R_l I^2 \tag{41}$$

The per unit efficiency of a *transmission* system is defined as the ratio of the output or load power to the power input to the transmission line. (Generator losses are not included in *transmission* efficiency.)

$$\text{transmission efficiency} = \frac{P_{ab}}{P_{a'b'}} = \frac{V_{ab} I}{V_{a'b'} I} = \frac{V_{ab}}{V_{a'b'}} \tag{42}$$

The per unit voltage regulation of a d-c transmission line is defined as the ratio of the difference between input and output voltages to the output voltage; that is,

$$\text{transmission line voltage regulation} = \frac{V_{a'b'} - V_{ab}}{V_{ab}} \tag{43}$$

Example. In Fig. 11, let it be assumed that $E = 110$ volts, $R_g = 0.1$ ohm, $R_l = 0.9$ ohm, and that R_L is so adjusted that the load voltage $V_{ab} = 100$ volts. The *efficiency of transmission* and the *voltage regulation of the transmission line* are to be found.

$$I = \frac{E - V_{ab}}{R_{gl}} = \frac{110 - 100}{0.1 + 0.9} = \frac{10}{1} = 10 \text{ amp}$$

$$\text{power delivered to the load } P_{ab} = 100 \times 10 = 1000 \text{ watts}$$

$$\begin{aligned}\text{transmission line input power } P_{a'b'} &= (E - R_g I) I \\ &= (110 - 1) \times 10 = 1090 \text{ watts}\end{aligned}$$

$$\textit{efficiency of transmission} = \frac{1000}{1090} = 0.918$$

$$\text{transmission line input voltage } V_{a'b'} = E - R_g I = 110 - 1 = 109 \text{ volts}$$

$$\textit{voltage regulation of the transmission line} = \frac{109 - 100}{100} = 0.09$$

voltage regulation of the overall system, including both the transmission

$$\text{line and generator voltage drops} = \frac{110 - 100}{100} = 0.10$$

13. Maximum Power Transfer. Refer to Fig. 11. Under the conditions which have been assumed, E, R_g, and R_l are constants and R_L is variable. [$(R_g + R_l) = R_{gl}$.] If R_L is infinitely large, as in an open circuit at the load end of the line, I equals zero and of course no power is delivered to the load. If R_L is reduced to zero, as in a short circuit at the load terminals ab, the power delivered to the load is equal to zero because $V_{ab} = R_L I = 0$.

For any finite value of R_L, the power delivered to the load is a finite value which is equal to $V_{ab}I$, in which V_{ab} equals $R_L I$.
Hence,

$$P_{ab} = V_{ab}I = R_L I^2 \tag{44}$$

in which

$$I = \frac{E}{R_{gl} + R_L}$$

Thus,

$$P_{ab} = E^2 \left[\frac{R_L}{(R_{gl} + R_L)^2} \right] \tag{45}$$

In order to find the relationship between the resistances (R_{gl} and R_L) which will make P_{ab} a maximum, P_{ab} may be differentiated with respect to the variable R_L and the derivative dP_{ab}/dR_L set equal to zero, as shown below:

$$\frac{dP_{ab}}{dR_L} = E^2 \left[\frac{(R_{gl} + R_L)^2 - 2R_L(R_{gl} + R_L)}{(R_{gl} + R_L)^4} \right] = 0$$

It follows that

$$(R_{gl} + R_L)^2 = 2R_L(R_{gl} + R_L) \tag{46}$$

which indicates that, for maximum power delivered to the load

$$R_L = R_{gl} = R_g + R_l \tag{47}$$

The power delivered to the load through a fixed generator and line resistance is maximum when the load resistance is adjusted to equal the sum of the generator and line resistances.

In communication circuits where the generator might take the form of a telphone transmitter or a radio microphone, very low power levels are encountered, and every effort is made to transmit the maximum power possible to the load or receiver. In low-power circuits, therefore, it is customary to *match* the load or receiver resistance to the line and generator resistance, that is, make $R_L = R_{gl}$.

In power circuits (meaning high-power circuits), no attempt is made to deliver maximum power to the load because the generators in this case are usually powerful devices which supply many parallel-connected loads. R_L is never reduced to a point where it even approximates the combined resistance of the generator and connecting lines because to do so would result in efficiencies of transmission of about 50 per cent. Moreover, most power devices are designed to operate at essentially constant voltage and, if the conditions stated in equation (47) were satisfied, the load voltage would vary from E volts at no load to half this value at maximum power transfer. Since a variation of only a few per cent in load voltage is permissible in practice, no attempt is made to match load and line resistances in power circuits. The load resistance is always many times the combined line and generator resistance in power circuits.

14. Attenuation—Design of Attenuation Sections. In communication circuits it is sometimes necessary to insert a four-terminal network into a two-wire line for the purpose of lowering the output power. A case like this might arise where a telephone is only a few hundred feet from the exchange battery which energizes not only this telephone but others which are miles from the exchange. For uniform operation of all the telephones, the line to the nearby telephone will have placed in it attenuation sections as shown in Fig. 12.

The balanced type of section shown in Fig. 12 may be rearranged for purposes of this analysis as a T-section of the kind shown in Fig. 13. The power output and the power input to this section may be written

$$P_{\text{out}} = V_{\text{out}} I_{\text{out}} \quad \text{and} \quad P_{\text{in}} = V_{\text{in}} I_{\text{in}}$$

Where four-terminal sections are inserted into communication lines, the ratio of P_{out} to P_{in} is usually more important than the actual values of these powers. The effect of the insertion is generally expressed in terms of a unit of attenuation called the *decibel*, abbreviated *db*. The meaning of this dimensionless unit of attenuation as applied to the present case is contained in the following equation:

$$db = 10 \log \frac{P_{\text{out}}}{P_{\text{in}}} = \text{(number of decibels resulting from insertion of a four-terminal network)} \tag{48}$$

Note: The logarithm to the base 10 is implied by log, since in this text $\log_{10}$ is written simply as *log*, and if logarithms to the base ε are intended the symbol *ln* is employed.

The relationship stated in equation (48) is used in a wide variety of ways in practice and is often applied to cases even where no four-terminal network is involved. For present purposes, however, the concept of a decibel as applied to a four-terminal network insertion is of

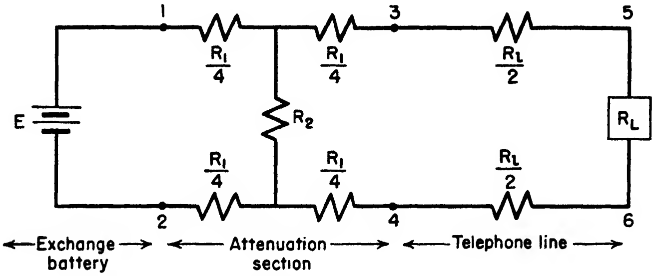


FIG. 12. Illustrating the use of an attenuation pad.

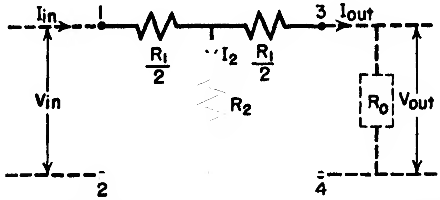


FIG. 13. Attenuation section in the form of a T-section.

immediate importance. If, for example, the four-terminal network were a power amplifier, P_{out} might be 100 times greater than P_{in}, in which case

$$db = 10 \log \frac{100}{1} = 10 \times 2 = +20 \quad \text{(decibel gain of amplifier)}$$

In the case of an attenuation section, P_{out} is obviously less than P_{in} because power (RI^2) is lost in the resistive elements which go to form the section. If, for example, $P_{\text{out}} = 0.1P_{\text{in}}$,

$$db = 10 \log \tfrac{1}{10} = 10(\log 1 - \log 10) = -10 \quad \text{(decibel loss)}$$

In using equation (48), a minus number of decibels indicates transmission loss (that is, attenuation), whereas a positive number indicates a transmission gain.

Returning now to the network shown in Fig. 13, the problem is so to design the section as to make R_{in} ($=V_{in}/I_{in}$) equal to R_{out} ($=V_{out}/I_{out}$) and at the same time make the section yield a specified number of decibels of attenuation. The total series-arm resistance R_1 of the section and the total shunt-arm resistance R_2 may be so adjusted as to give the desired results.

Letting the output resistance be symbolized as R_o, it is plain that

$$R_{in} = \frac{R_1}{2} + \frac{\left(\frac{R_1}{2} + R_o\right)R_2}{\frac{R_1}{2} + R_2 + R_o} \quad \begin{array}{l}\text{(resistance looking}\\ \text{into terminals 1-2}\\ \text{of Fig. 13)}\end{array} \tag{49}$$

and if we *make* the above value of R_{in} equal to R_o we have

$$R_o\left(\frac{R_1}{2} + R_2 + R_o\right) = \frac{R_1}{2}\left(\frac{R_1}{2} + R_2 + R_o\right) + \left(\frac{R_1}{2} + R_o\right)R_2 \tag{49-a}$$

Solving for R_o gives

$$R_o = \sqrt{R_1R_2 + \frac{R_1^2}{4}} \tag{50}$$

The importance of equation (50) is that we can make the equivalent resistance looking to terminals 1 and 2 of Fig. 13 precisely the same as the load resistance by arbitrarily selecting any R_1 and solving for R_2 in terms of the specified R_o and the arbitrarily selected R_1. In the design of attenuation sections, however, we do not arbitrarily select R_1; rather we impose the condition that R_1 and R_2 shall be selected to yield a specified number of decibels of transmission loss and at the same time make R_{in} equal to R_o.

If $R_{in} = R_o$,

$$n = \text{(No. of)db} = 10 \log \frac{V_{out}I_{out}}{V_{in}I_{in}} = 10 \log \frac{I_{out}^2R_o}{I_{in}^2R_{in}}$$

or

$$n = 20 \log \frac{I_{out}}{I_{in}}$$

Solving the above relationship for the ratio I_{out}/I_{in},

$$\frac{I_{out}}{I_{in}} = 10^{n/20} = K \tag{51}$$

Thus if n is specified as some negative number of decibels, I_{out}/I_{in} is a fixed quantity which is here symbolized as K.

By ordinary circuit analysis, it follows that

$$I_{\text{out}} = I_{\text{in}} \left[\frac{R_2\left(\frac{R_1}{2} + R_o\right)}{\frac{R_1}{2} + R_2 + R_o} \right] \frac{1}{\left(\frac{R_1}{2} + R_o\right)}$$

The ratio $I_{\text{out}}/I_{\text{in}}$ then reduces to simply

$$K = \frac{I_{\text{out}}}{I_{\text{in}}} = \frac{R_2}{\frac{R_1}{2} + R_2 + R_o} \tag{52}$$

From equation (50) it is known that

$$R_1 R_2 + \frac{R_1^2}{4} = R_o^2 \tag{53}$$

If R_o and K are specified quantities in equations (52) and (53), these equations may be solved simultaneously for R_1 and R_2. The results may be stated in a variety of different algebraic forms, the simplest of which are probably

$$R_2 = \frac{2R_o K}{1 - K^2} \tag{54-a}$$

$$R_1 = 2R_o \frac{(1 - K)}{(1 + K)} \tag{54-b}$$

Example. Let it be required to find R_1 and R_2 in Fig. 13 if

$$R_o = 100 \quad \text{ohms}$$

and an attenuation of 5 db is desired.

From equation (51),

$$K = 10^{-5/20} = 0.563$$

Then, from equations (54-a) and (54-b),

$$R_2 = \frac{200(0.563)}{1 - 0.563^2} = 164.6 \quad \text{ohms}$$

$$R_1 = 200 \frac{(1 - 0.563)}{(1 + 0.563)} = 56 \quad \text{ohms}$$

The required attenuation section takes the form shown in Fig. 14 since R_1 (the total series-arm resistance) is equally divided on either side of R_2.

Exercise. Show by ordinary methods that the equivalent series-circuit resistance looking into terminals 1 and 2 of Fig. 14 is 100 ohms.

15. Double-Subscript Notation. At various places throughout the text, double-subscript notation has been used either where the *order* of the subscripts was obvious from physical considerations, or where the *order* of the subscripts was immaterial. The symbol R_{ab} in Fig. 15, for

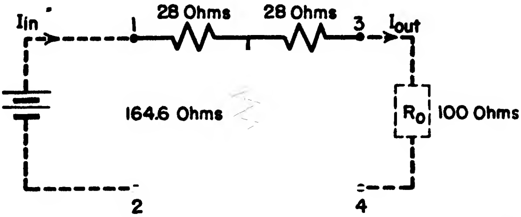


FIG. 14. Section which yields 5 db attenuation in a 100-ohm line.

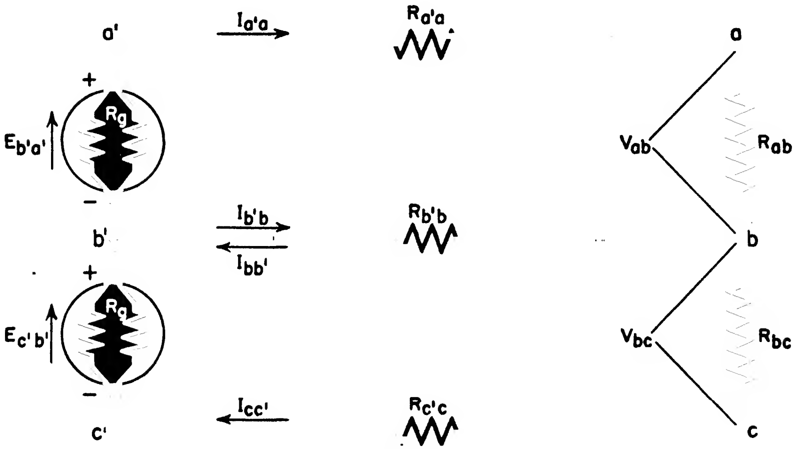


FIG. 15. A three-wire transmission system used for illustrating double-subscript notation.

example, means simply the resistance between points a and b of the circuit. In cases of this kind, the *order* or *sense* of the subscripts is immaterial since $R_{ba} = R_{ab}$.

When, however, double subscripts are attached to currents and voltages, they are usually intended to indicate the *sense* or *direction* in which the currents or voltages are being considered. In single-subscript notation an arrow accomplishes the same purpose. Whether we use arrows on a circuit diagram to indicate the intended arrow direction, or whether we indicate this arrow direction by some system of subscripts is a matter of choice. If the theory underlying *arrow directions* is properly understood, no confusion should arise, even though some of

the arrow directions are indicated by arrows on the circuit diagram and some of them are indicated by double subscripts attached to the current and voltage symbols. A mixture of the two systems is not recommended, however.

The order in which the subscripts are written indicates clearly the intended arrow direction. In Fig. 15, for example, $I_{b'b}$ (without an arrow) means that we are taking the arrow direction of the current in this branch *from b' to b*. $I_{bb'}$ means the current *from b to b'* is intended as the arrow direction. $I_{bb'} = -I_{b'b}$. It has been shown elsewhere in this chapter that an arrow direction (as indicated by either an arrow or double subscript) does not imply a knowledge of the actual positive direction of current flow. The arrow direction is used simply as an artifice in obtaining a consistent set of signs during the formation of the basic circuit equations. If the *assumed* arrow-directed current turns out to be a negative quantity, the correct positive direction of the actual current is opposite to the *assumed* direction.

Example. Let it be required to establish the Kirchhoff law equations for the network shown in Fig. 15 by using a consistent set of double-subscript notation.

Current law at junction b: $\quad I_{a'a} = I_{bb'} + I_{cc'}$

(Having selected the currents $I_{a'a}$, $I_{bb'}$, and $I_{cc'}$ as the arrow directions for the three branch currents, it is well to use these same currents in the remaining equations.)

Voltage law around upper loop:

$$(R_{a'a} + R_{ab} + R_g)I_{a'a} + R_{bb'}I_{bb'} = E_{b'a'}$$

Voltage law around lower loop:

$$(R_{c'c} + R_{bc} + R_g)I_{cc'} - R_{bb'}I_{bb'} = E_{c'b'}$$

In writing the two voltage equations it has been assumed that the generator resistances are equal in magnitude. If this were not the case, the resistance of the $E_{b'a'}$ generator could be written as $R_{b'a'}$; the other as $R_{c'b'}$.

The network shown in Fig. 15 is sometimes used for the transmission of 115/230 volts, direct current, and is usually referred to as a three-wire system. In this case the generators are similar in construction and are sometimes built into the same rotating machine. The line $b'b$ is called the *neutral* because, if $R_{ab} = R_{bc}$, the upper and lower loops of Fig. 15 are then symmetrical and no potential difference appears between the points b' and b. A corresponding three-wire system is widely used in the distribution of 115/230 volts, alternating current, such as is supplied to dwellings. In the a-c three-wire system the generators

shown in Fig. 15 are replaced by two secondary windings of a transformer connected together at point b' and the neutral wire $b'b$ is grounded in the interests of safety.

EXTENSIONS TO BASIC METHODS OF CIRCUIT ANALYSIS

16. Nodal Method of Analysis. The method of analysis thus far employed in this text has been that of writing voltage equations which are dictated by the voltage equilibrium conditions that must exist around the closed loops of the network. A relatively new method of analysis accomplishes the same results by means of current equations which are dictated by the current equilibrium conditions at the junctions (or nodes) of the network, that is, by Kirchhoff's current law. The nodal method of analysis uses well-known principles and has been used successfully in both transient analysis[4] and vacuum-tube circuit analysis.[5] A considerable reduction in labor can be effected by the use of the nodal method in many network configurations.

In its simplest sense, a *node* of a network is any accessible terminal which is at a significant potential difference with respect to the other terminals. In this sense, the network shown in Fig. 16-a *might be* considered as a four-node network. Only the junction points of a network, however, *need be* considered as nodes since the number of independent nodes is the number of junction points minus one. This will become more evident as we proceed. Before the nodal method of analysis can be employed, the specified *voltage* sources must be transformed to equivalent *current* sources in accordance with the principles contained in the following section.

17. Transformation from a Voltage Source to an Equivalent Current Source. It will be shown presently that currents rather than voltages are used as the specified quantities (or independent variables) in the nodal method of analysis. The voltage sources which are usually specified in the original network are therefore replaced by equivalent current sources. If, for example, the voltage source E_1 in Fig. 16-a were a specified voltage, the equivalent current source might be found by noting that

$$I_x = \frac{(V_1 - V_4) - (V_2 - V_4)}{R_1} \tag{55}$$

and that, since $V_1 = E_1$,

$$I_x = \frac{V_1 - V_2}{R_1} = \frac{V_1}{R_1} - \frac{V_2}{R_1} = \frac{E_1}{R_1} - \frac{V_2}{R_1} \tag{56}$$

[4] *Transients in Linear Systems*, Gardner and Barnes, John Wiley and Sons.
[5] *Network Analysis and Feedback Amplifier Design*, Bode, D. Van Nostrand.

The purpose of writing $V_1 - V_4$ and $V_2 - V_4$ in equation (55) is to emphasize the fact that the potential of each node is actually a potential difference referred to some node as reference. In many cases this reference node can be reckoned as zero, particularly where it is grounded physically or where it is a common chassis connection as in radio sets.

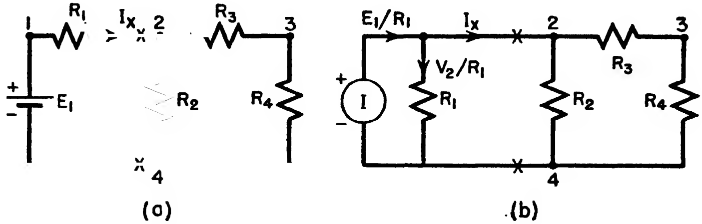


FIG. 16. Illustrating the transformation of a voltage source E_1, in (*a*), to an equivalent current source I, in (*b*).

Equation (56) indicates that the current entering node 2 of Fig. 16-a is a fixed and specified current (E_1/R_1) minus another current, the value of which is V_2/R_1. The voltage V_2 is not known initially but, as will be shown presently, it enters the equations as a dependent variable. If E_1 and R_1 (to the left of XX in Fig. 16-a) are replaced with a configuration like that shown in Fig. 16-b, I_x, V_2, and the rest of the currents and voltages will be the same as in the original circuit, namely, Fig. 16-a. The reason is that I_x and V_2 remain precisely the same in the two cases and the circuit arrangement to the right of XX has not been changed. Another way of stating the reason for the equivalence of Fig. 16-a and Fig. 16-b is to say that I_x and V_2 remain invariant during the transformation from the voltage source, E_1 of Fig. 16-a, to the current source I of Fig. 16-b.

Example. In Fig. 16-a, let $E_1 = 10$ volts, $R_1 = 1$, $R_2 = R_3 = 2$, and $R_4 = 5$ ohms. Let it be required to find V_3, the output voltage of the network.

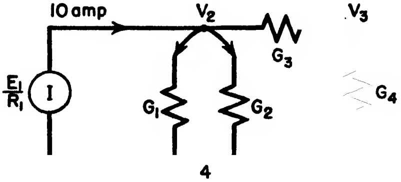


FIG. 17. Application of the nodal method to a simple network.

In this type of analysis, conductances are usually more convenient to use than are the resistances; therefore R_1 is replaced by G_1, R_2 by G_2, and so on.

After the voltage source is changed to a current source the configuration shown in Fig. 17 is obtained. This is a two-node network effectively, if voltages are reckoned from node 4. In other words, the two unknown voltages of the system are the voltages at node 2 and node 3. The relationships necessary for evaluating these voltages

are easily established by writing Kirchhoff's current law first at node 2 and then at node 3. It will be observed that, whereas node 2 is a junction in the sense that the current divides there, node 3 is not a junction. Writing I_{away} from a node $= I_{\text{toward}}$ a node in Fig. 17 gives:

$$\text{(At node 2):} \qquad G_1V_2 + G_2V_2 + G_3(V_2 - V_3) = 10 \quad \text{(amp)}$$

$$\text{(At node 3):} \qquad G_4V_3 + G_3(V_3 - V_2) = 0$$

Or

$$(G_1 + G_2 + G_3)V_2 - G_3V_3 = 10$$

and

$$-G_3V_2 + (G_3 + G_4)V_3 = 0$$

Numerically, $G_1 = 1$, $G_2 = G_3 = 0.5$, and $G_4 = 0.2$ mho. Hence,

$$V_3 = \frac{\begin{vmatrix} 2 & 10 \\ -0.5 & 0 \end{vmatrix}}{\begin{vmatrix} 2 & -0.5 \\ -0.5 & 0.7 \end{vmatrix}} = \frac{5}{1.15} = 4.348 \quad \text{volts}$$

No particular advantage results from the use of the nodal method in this simple case because V_3 can be found quite simply by other methods.

18. Application of the Nodal Method. The example given above shows that Kirchhoff's current law is used to form the basic equations in the nodal method in much the same way as the voltage law is used to form the basic equations in the mesh-current method. In the nodal method, no general equations are necessary since, in an N-node network, the application of the current law to $N - 1$ nodes will yield the required number of independent relationships. The current equations can usually be written very quickly from an inspection of the circuit configuration.

In the writing of current equations as the basic relationships, the voltages of the various nodes become the dependent variables or sought-after quantities. If the voltage of each node of a network is known, any current in the network can readily be found. In the mesh-current method, the currents are solved for and then the required voltages calculated. In the nodal method, the voltages are solved for and then the required currents. The nodal method of analysis can be used advantageously in many vacuum-tube circuits and in other configurations where the number of meshes is greater than the number of nodes.

By way of further illustration, the network shown in Fig. 18 will be solved by the nodal method for V_2. Configurations of this kind are frequently encountered after a vacuum-tube element in the actual circuit has been replaced by the equivalent R_p branch shown in Fig. 18. (The actual transformation of the vacuum-tube element will not be

shown at this stage because the voltages involved are usually a-c voltages. See Chapter XV.) For the present, the voltage sources shown in Fig. 18 are assumed to be batteries having open-circuit voltages as specified.

The E_1 source of Fig. 18 is replaced with an E_1/R_1 current source exactly as in Fig. 16. The E_x source is replaced by an equivalent current

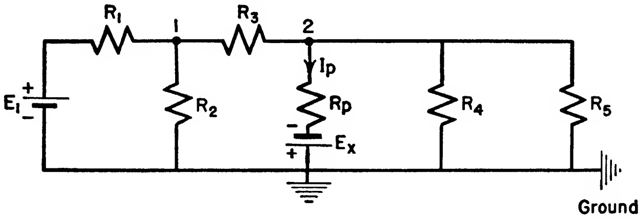


FIG. 18. A network which can be reduced to a 2-node network by replacing the voltage sources with equivalent current sources.

source in much the same manner. The details of this transformation are given below. (Refer to Fig. 19.)

(1) The current flowing through R_p (or G_p) and the E_x source (in the direction from node 2 to ground) is

$$I_p = [V_2 - (-E_x)]\,G_p = V_2G_p + E_xG_p \tag{57}$$

where V_2 is the voltage of node 2 relative to ground.

(2) If E_x is now removed from the circuit, V_2G_p will be the current flowing from node 2 to ground through G_p; therefore G_p occupies the position in the transformed circuit indicated in Fig. 19-b.

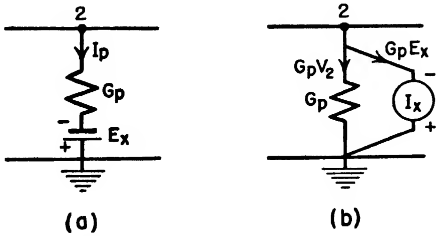


FIG. 19. Transformation of the E_x voltage source shown in Fig. 18 to an equivalent current source; $G_pE_x = I_x$.

(3) Equation (57) indicates that, under the transformation proposed, another current ($G_pE_x = I_x$) flows away from node 2. I_x is a known current if G_p and E_x are specified. In this transformation it will be observed that the voltage of node 2 and the current flowing away from this node remain unchanged.

The arrangement given in Fig. 20 shows how the two voltage sources

of Fig. 18 may be replaced by two current sources. Equivalent current sources of this kind are sometimes referred to as infinite resistance sources since no variation of the network resistances (except those used

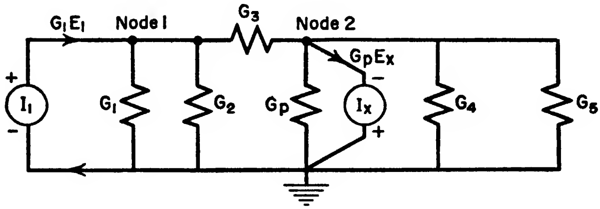


FIG. 20. The equivalent of the network shown in Fig. 18.

to define the current sources) can affect the magnitudes of the current sources. Again writing I (away from a node) = I (toward a node) in Fig. 20:

At node 1:

$$(G_1 + G_2)V_1 + G_3(V_1 - V_2) = G_1E_1 = I_1 \tag{58}$$

At node 2:

$$(G_p + G_4 + G_5)V_2 + G_3(V_2 - V_1) = -G_pE_x = -I_x \tag{59}$$

Since I_1 and I_x are known currents they are written as the right-hand members of the current equations.

Equations (58) and (59) may be written more systematically as

$$(G_1 + G_2 + G_3)V_1 - G_3V_2 = I_1 \tag{58-a}$$

$$-G_3V_1 + (G_3 + G_4 + G_5 + G_p)V_2 = -I_x \tag{59-a}$$

Example. In the network shown in Fig. 18

$E_1 = 1$ volt, $E_x = 10$ volts, $R_1 = R_2 = R_3 = R_4 = R_5 = R_p = 1$ ohm

Let it be required to find the voltage of node 2 (V_2) relative to ground.

Replacing the voltage sources with equivalent current sources as indicated in Fig. 20 gives

$$I_1 = G_1E_1 = 1 \text{ amp} \quad \text{and} \quad I_x = G_pE_x = 10 \text{ amp}$$

From equations (58-a) and (59-a),

$$3V_1 - 1V_2 = 1$$
$$-1V_1 + 4V_2 = -10$$

Solving for V_2:

$$V_2 = \frac{\begin{vmatrix} 3 & 1 \\ -1 & -10 \end{vmatrix}}{\begin{vmatrix} 3 & -1 \\ -1 & 4 \end{vmatrix}} = -\frac{29}{11} \text{ volts}$$

Thus node 2 is shown to be 29/11 volts below the potential of the ground bus.

The solution to this problem might have been carried out by the mesh-current method with the use of the specified voltage sources. After the R_4 and R_5 branches have been reduced to an equivalent resistance, however, a three-row three-column determinant solution will be required.

The nodal method of analysis is usually superior to the mesh-current method if the number of nodes (after transformation to current sources) does not exceed the number of meshes or loops. If N represents the number of nodes in a network, only $N - 1$ independent node equations are required, and these are obtained by applying Kirchhoff's current law to $N - 1$ nodes. In some types of networks, the choice of the nodal method will change the problem from a four-row four-column determinant type to a two-row two-column type, and in cases of this kind a considerable amount of labor is saved if the nodal method is used. (See Prob. 30 at the close of the chapter.) The full significance of the power of the nodal method will probably not be appreciated until after networks which contain vacuum tubes have been encountered. Networks of this kind are studied in detail in engineering electronics courses.

Before the discussion of mesh-current and nodal methods of circuit analysis is ended, it should be pointed out that both methods are applicable to a-c circuits as well as to d-c circuits. If alternating voltages are employed to energize the network, the R's in equations (34) are replaced by impedances (to alternating current). Impedances are functions of frequency and are symbolized by the letter Z. The reciprocals of impedances are called admittances and are symbolized by Y.

In a-c circuit analysis the G's of equations (58) and (59) would be replaced by Y's, and, in a general N-junction network, the $N - 1 = n$ independent current equations would take the form

$$\left.\begin{array}{l} Y_{11}V_1 - Y_{12}V_2 - Y_{13}V_3 - \cdots\cdots - Y_{1n}V_n = I_1 \\ -Y_{21}V_1 + Y_{22}V_2 - Y_{23}V_3 - \cdots\cdots - Y_{2n}V_n = I_2 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ -Y_{n1}V_1 - Y_{n2}V_2 - Y_{n3}V_3 - \cdots\cdots + Y_{nn}V_n = I_n \end{array}\right\} \begin{array}{l}\text{General} \\ \text{System of} \\ \text{Nodal} \\ \text{Equations}\end{array}$$

The self-admittances (or conductances) Y_{11}, $Y_{22}, \cdots$, Y_{nn} are respectively *the admittances (or conductances) of each of the nodes relative to all the other nodes or junctions.* In the general system of equations given above, the nodes are selected as the junctions of the network.

The mutual admittances (or conductances) Y_{12}, Y_{13}, Y_{21}, Y_{23}, and the like are plainly the admittances (or conductances) which exist between the two junctions designated by the subscripts. If linear bilateral circuit elements are involved, $Y_{nm} = Y_{mn}$.

With a little practice the determinant solutions for networks employ-

ing admittances (or conductances) may be established directly from an inspection of the network after the specified voltage generators have been transformed to equivalent current generators. The outputs of these current generators are represented in the above system of equations as I_1, I_2, and I_n.

19. Network Theorems. The application of network theorems often provides a deeper insight into the operation of the network than does a straight forward application of Kirchhoff's laws. In some instances these network theorems may be employed as labor-saving devices as well.

Probably the most widely used of the various network theorems is Thévenin's theorem which is discussed in some detail in Section 21. The proof of this important theorem rests upon the superposition principle outlined below.

20. The Superposition Principle as Applied to Circuit Analysis. The law of superposition has already been applied to the calculation of electric forces in Section 3 of Chapter II. This same principle or law may be employed to calculate the current in any part of a linear network due to several emf's by the simple procedure of calculating the component currents in this part of the circuit due to the various emf's, taken one at a time, and then adding the component currents algebraically.

The superposition principle as applied to circuit analysis is correct since each emf acting alone satisfies the Kirchhoff law relations of the network; hence the sum of all the solutions (for the various emf's) satisfies these relations.

As applied to elementary types of networks, the principle of superposition is of little practical importance since the current at any point in the network due to several emf's can usually be determined by methods which involve less work than would the application of the superposition principle. From a theoretical point of view, however, the superposition principle is of great importance in establishing other network theorems.

Example. Simply to illustrate the meaning of the superposition principle as applied to the calculation of a current due to two emf's, the current through the 2-ohm resistor shown in Fig. 21-a will be determined by means of this principle.

The current in question is actually 24/5 amp, as may be seen from the mesh-current solution

$$I = I_1(\text{in Fig. 21-a}) = \frac{\begin{vmatrix} 10 & -1 \\ 4 & 2 \end{vmatrix}}{\begin{vmatrix} 3 & -1 \\ -1 & 2 \end{vmatrix}} = \frac{24}{5} \text{ amp}$$

If the 4-volt battery is replaced with a resistance equal to its internal resist-

ance (which is here taken equal to zero),

$$I_{\text{2-ohm (due to 10-v emf)}} = \frac{10}{2.5} = \frac{20}{5} \text{ amp} \qquad \text{(See Fig. 21-b.)}$$

Then if the 10-volt battery is replaced with a short circuit as shown in Fig. 21-c,

$$I_{\text{2-ohm (due to 4-volt emf)}} = \frac{\frac{4}{1 + \frac{2}{3}} \times \frac{2}{3}}{2} = \frac{4}{5} \text{ amp}$$

The sum of the component currents equals the actual current in the 2-ohm resistor as predicted by the superposition principle.

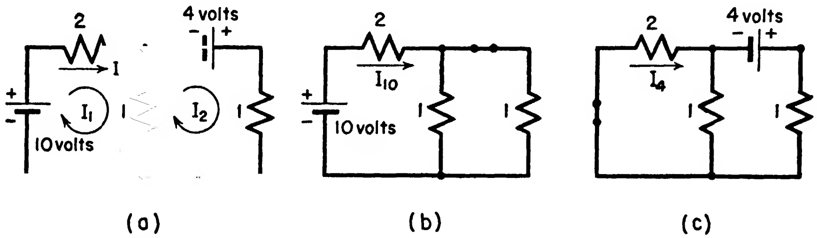


FIG. 21. Illustrating the superposition principle. (Numbers near resistor symbols indicate ohms.)

The chief precautions to observe in the use of the superposition principle are:

(1) Replace all emf's except the pertinent one with their internal resistances if these resistances are significantly large,

(2) Combine the component currents *algebraically*, that is, with due regard for the circuit direction of each component.

21. Thévenin's Theorem. Consider an active network composed of linear circuit elements and energized with any number of voltage sources or emf's. One form of Thévenin's theorem states that

Any two terminals of a network may be regarded as a single emf source having an internal resistance equal to the equivalent series-circuit resistance looking back into the network from these two terminals if the open-circuit voltage of the single emf source is taken as the voltage which appears across these two terminals before any additional load (or emf) is placed across these terminals.

In order to understand better the full meaning of this theorem, consider any linear-bilateral network which is energized with any number of emf's. Take *any* two terminals of this energized network whatsoever and let them be known as the specified terminals. *With respect to these specified terminals, the network may be replaced by a single emf, say E_t, in*

series with a single resistance, say R_t. E_t is the voltage which appears across the specified terminals of the original network. R_t is the resistance looking back into the passive network from the specified terminals when all emf sources have been replaced with their internal resistances.

The current that will flow in any *new* branch which is placed across the specified terminals will be

$$I = \frac{E_t + E_n}{R_t + R_n} \quad \text{(equational form of Thévenin's theorem)} \qquad (60)$$

where E_t and R_t have been previously defined
E_n is the emf of the new or added branch
R_n is the resistance of the new or added branch.

Although equation (60) has been developed on the basis of a new branch being added to the network, this new branch might actually be

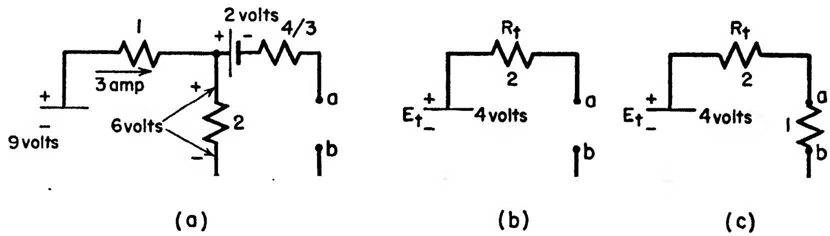


FIG. 22. Illustrating Thévenin's theorem for a resistance insertion.

any branch of the original network. In this case, equation (60) applies directly simply by opening the branch in question and treating the open ends as the specified terminals. The technique involved can best be illustrated by means of examples, and the proof of Thévenin's theorem can best be presented after the reader fully understands the technique employed in the application of the theorem.

Example 1. Consider terminals *ab* of Fig. 22-a with a view toward calculating the current that will flow through a 1-ohm resistor placed across the terminals *a* and *b*. If the active network is considered as a single emf source having an internal resistance in accordance with Thévenin's theorem:

$$\text{Thévenin-equivalent emf, } E_t = \frac{9}{3} \times 2 - 2 = 4 \quad \text{volts (terminal } a+)$$

$$\text{Thévenin-equivalent resistance, } R_t = \frac{4}{3} + \frac{2 \times 1}{3} = 2 \quad \text{ohms}$$

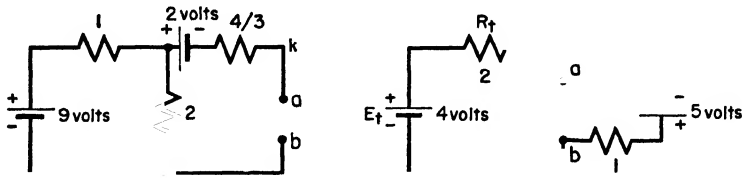
If a 1-ohm resistor is placed across terminals *a* and *b* of the active network,

Thévenin's theorem affirms that

$$I_{ab} = \frac{E_t}{R_t + R_{ab}} = \frac{4}{2+1} = \frac{4}{3} \quad \text{amp}$$

The reduction of the network shown in Fig. 22-a to its Thévenin-theorem equivalent is shown in Fig. 22-b. The result obtained in Fig. 22-c may be checked readily by a mesh-current solution of Fig. 22-a, if it is assumed that a 1-ohm resistor is placed across terminals a and b of Fig. 22-a. (Terminals a and b might actually be a break in the 2-volt branch of Fig. 22-a if this branch is assumed to have a resistance of $\frac{7}{3}$ ohms.)

Example 2. Nothing in the statement of Thévenin's theorem prevents the insertion of a voltage source across the two terminals of the active network. To illustrate this point let it be assumed that an emf of 5 volts having an internal resistance of 1 ohm is to be inserted across terminals a and b of Fig. 23-a with its negative terminal at terminal a. It is required to find the current



(a) (b)

FIG. 23. A Thévenin-theorem solution. (See Example 2.)

in the $\frac{4}{3}$-ohm resistor of the original network which results from this insertion. Since Fig. 23-a is the same as Fig. 22-a, the Thévenin-theorem solution is simply

$$I_{ab} = I_{(\text{in 4/3-ohm resistor})} = \frac{E_t + E_{ab}}{R_t + R_{ab}} = \frac{4+5}{2+1} = 3 \quad \text{amp}$$

The manner in which this calculation is carried out is shown in Fig. 23-b.

To check this result it is simply necessary to insert the 5-volt emf (with its 1-ohm internal resistance) across terminals a and b of Fig. 23-a and solve for the required current:

$$I_{(\text{4/3-ohm resistor})} = I_2 = \frac{\begin{vmatrix} 3 & 9 \\ -2 & 3 \end{vmatrix}}{\begin{vmatrix} 3 & -2 \\ -2 & \frac{13}{3} \end{vmatrix}} = \frac{9+18}{13-4} = 3 \quad \text{amp}$$

where I_2 is the loop current around the right-hand loop of the network.

Example 3. In examples 1 and 2, terminals a and b were originally open-circuited. This condition, however, is not required for the application of Thévenin's theorem. Consider terminals a and b of Fig. 24-a with a view toward calculating the current that will flow through a 0.7-ohm resistor placed across terminals a and b.

A mesh-current solution for I_2 in the original circuit will be used to find the Thévenin-equivalent emf across terminals a and b. As indicated in Fig. 24-a,

$$\text{Thévenin-equivalent emf, } E_t = 1 \times I_2 = 1 \times \frac{\begin{vmatrix} 3 & 10 \\ -1 & 4 \end{vmatrix}}{\begin{vmatrix} 3 & -1 \\ -1 & 2 \end{vmatrix}} = 4.4 \quad \text{volts}$$

The Thévenin-equivalent resistance, that is, the resistance looking back into the network from the a and b terminals, is observed to be the resistance of three parallel branches: 1 ohm, 1 ohm, and 2 ohms in parallel. Thus,

$$\text{Thévenin-equivalent resistance, } R_t = \frac{0.5 \times 2}{2.5} = 0.4 \quad \text{ohm}$$

$$I_{ab\,(\text{through 0.7-ohm resistor})} = \frac{E_t}{R_t + 0.7} = \frac{4.4}{0.4 + 0.7} = 4 \quad \text{amp}$$

This result may be checked by a mesh-current solution of Fig. 24-a if it is assumed that the 0.7-ohm resistor is placed across terminals a and b.

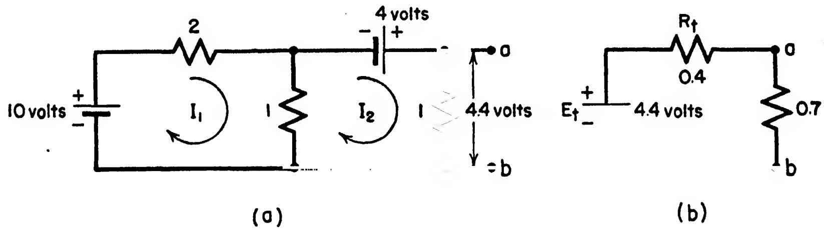


FIG. 24. A Thévenin-theorem solution. (See Example 3.)

22. Proof of Thévenin's Theorem. Thévenin's theorem is essentially a specialized or restricted form of the superposition principle, as will be shown presently. The proof of Thévenin's theorem may be stated most simply in terms of an opening in any branch of the network as, for example, the break at $a - b$ in Fig. 23-a. [As applied to equation (60), this means that we shall consider first the case where E_n and R_n are both zero.]

A generalized proof of Thévenin's theorem requires several detailed steps, each of which will be stated in general terms and then illustrated in so far as these detailed steps apply to the evaluation of the current in branch k of Fig. 23-a; it is assumed that points a and b of this branch are originally joined together to complete branch k. A mesh-current solution will show that the current flowing from k to a in this branch (due to the two original emf sources) is 2 amp, and this numerical value will be used as an example of what is meant by I_k in the generalized proof.

(1) Upon opening any branch (say the kth branch of the network) an open-circuit voltage E_t will appear across the break and the current in the kth branch, say I_k, will of course be zero. (As applied to Fig. 23-a, this means that, if we open branch k, I_k will be zero and a voltage of 4 volts will appear across terminals a and b with terminal a positive.)

(2) Now suppose that a fictitious voltage $-E_t$ of exactly the magnitude of the open-circuit voltage is introduced across the break. This, obviously, will not change anything from the open-circuit condition in so far as the rest of the network is concerned provided the fictitious voltage $-E_t$ is introduced across the break with the polarity which maintains $I_k = 0$.

(As applied to Fig. 23-a, this means simply that a 4-volt emf source is introduced across terminals a and b with the positive terminal at a. Under these conditions a Kirchhoff law solution of the resulting network will show that the current in branch k is zero.)

(3) There is now a network with no open branches but with a new voltage source equal to $-E_t$ introduced into the network. Under this new condition, the branch current I_k is zero and, according to the superposition principle, the current distribution resulting in the network from the introduction of $-E_t$ with all other voltage sources short-circuited must be such as to cancel exactly the original current in branch k. The current flowing originally in the kth branch therefore must be equal in magnitude but opposite in direction to that established by $-E_t$.

(As applied to Fig. 23-a, this means that $-E_t$ establishes a current in branch k which is 2 amp in magnitude and directed from a to k since as previously shown the 9-volt and 2-volt emf sources establish a current which is equal to 2 amp directed from k to a. This condition must of course exist if $I_k = 0$ with $-E_t$ inserted across terminals a and b.)

(4) If all the original emf sources were short-circuited and only $-E_t$ were operative, the current in the kth branch could be calculated as $-E_t/R_t$, where R_t is the resistance looking back into the network from the terminals of the fictitious emf source $-E_t$.

[As applied to Fig. 23-a, this means that $-E_t$ (= 4 volts) establishes a current equal to 2 amp in branch k which is directed from a to k since with the original emf sources short-circuited $-E_t$ sees 2 ohms looking back into the network.]

(5) Since the current in branch k due to the original emf sources is desired, it is simply necessary to reverse the polarity of the fictitious emf ($-E_t$) in a simple series circuit which contains a resistance equal to R_t and determine the current due to the original emf sources as E_t/R_t.

(In Fig. 23-a, this current will be 2 amp directed from k to a.)

Example. Let the 1-ohm resistor branch in Fig. 25-a be the kth branch of the network. It is required to find the current in the 1-ohm resistor in accordance with steps outlined above.

(1) Break the 1-ohm branch at x–x' and determine by ordinary circuit analysis that

$$V_{xx'} = 4 \text{ volts} \quad \text{(with the } x \text{ terminal positive)}$$

(2) Now put $-E_t = 4$ volts across the break with the positive end at the x terminal as shown in Fig. 25-b.

(3) In the circuit arrangement shown in Fig. 25-b,

$$I_k = I_k(\text{due to } -E_t) + I_k(\text{due to other emf's}) = 0$$

(4)

$$I_{k(\text{due to } -E_t)} = \frac{-E_t}{R_t} \quad \text{(directed in the } -I_k \text{ direction)}$$

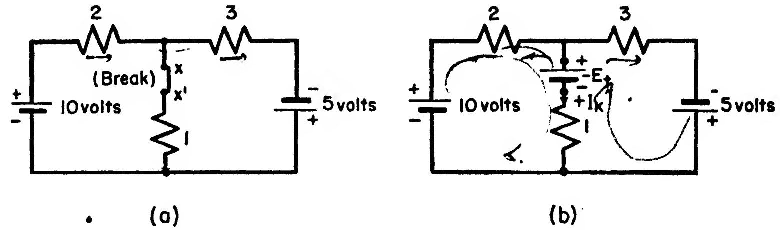


FIG. 25. In (a) the open-circuit voltage across x–x' is 4 volts, and in (b) it is $I_k = 0$.

where R_t is the resistance which $-E_t$, shown in Fig. 25-b, sees looking back into the network. Thus,

$$R_t = 1 + \frac{2 \times 3}{2 + 3} = \frac{11}{5} \quad \text{ohms}$$

(5)

$$I_k = \frac{4}{\frac{11}{5}} = \frac{20}{11} \quad \text{amp} \quad \text{(directed in the } +I_k \text{ direction)}$$

The equivalent open-circuit voltage E_t and the equivalent internal resistance of the network R_t are quantities which can be readily measured in the laboratory. E_t may be measured with a high-resistance voltmeter across the break, and R_t may be measured as E_t/I_{sc} where I_{sc} is the current which is measured by a low-resistance ammeter placed across the break.

The proof[6] of Thévenin's theorem given above may readily be extended to include additional emf's and resistances placed *across* the break simply by recognizing that this theorem first reduces the network to an equivalent series circuit where an additional emf or resistance can be accounted for as shown in equation (60).

[6] This proof is essentially the same as that given by Walther Richter in "Applications of Thévenin's Theorem," *Electrical Engineering*, March 1945, pp. 103–105.

PROBLEMS

1. What is the resistance looking into terminals *a* and *b* of Fig. 26, that is, the equivalent series-circuit resistance between these terminals?

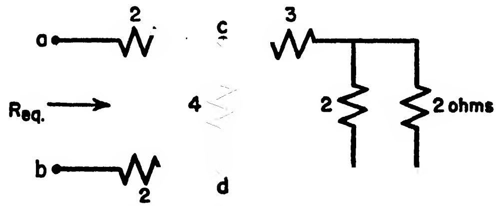


FIG. 26. See Probs. 1 and 2. (Numbers indicate ohms.)

2. What is the conductance between points *a* and *d* of Fig. 26 if terminals *a* and *b* are connected together?

3. Transform the delta-connected resistors shown in Fig. 27 to an equivalent set of wye-connected resistors between terminals *a*, *b*, and *c*.

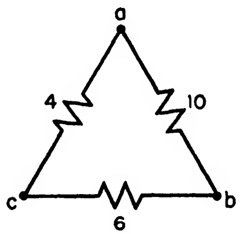


FIG. 27. See Prob. 3.

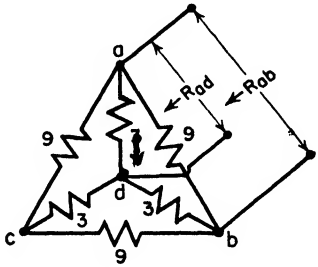


FIG. 28. See Probs. 4 and 5.

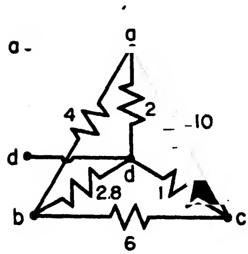


FIG. 29. See Prob. 6. (Numbers indicate ohms.)

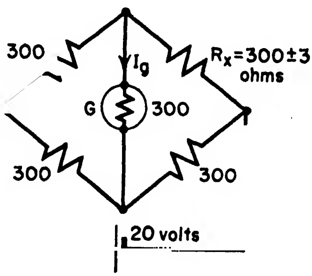


FIG. 30. See Probs. 8 and 34.

4. What is the resistance looking into terminals *a* and *b* of Fig. 28?

5. What is the resistance looking into terminals *a* and *d* of Fig. 28?

6. What is the resistance looking into terminals *a* and *d* of Fig. 29?

7. A set of twelve 2-ohm resistors are arranged to form a cube. What is the resistance between diagonally opposite corners of this cube? *Note*: Take advantage of the symmetry of this arrangement.

8. The Wheatstone bridge arrangement shown in Fig. 30 is to be used to test 1 per cent tolerance resistors which are to be 300 ± 3 ohms. The bridge has been balanced so that $I_g = 0$ when R_x is precisely 300 ohms. The galvanometer is a zero-center instrument which reads ± 500 microamp full-scale. What is the maximum + or − reading of the galvanometer for a resistor to be acceptable? (The internal resistance of the battery is negligibly small, and it is assumed that the difference between a + and a − maximum is negligible.)

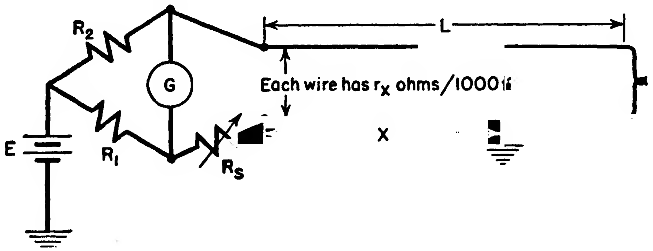


FIG. 31. Varley loop for measuring distance X. (See Prob. 9.)

9. The Varley loop shown in Fig. 31 is often used in measuring the distance out to a "ground" on a telephone line.

(*a*) If each of the line wires has r_x ohms/1000 ft show that, under balanced bridge conditions,

$$X = \frac{2L\dfrac{R_1}{R_2} - \dfrac{R_s}{r_x}}{1 + \dfrac{R_1}{R_2}} \quad \text{thousands of feet}$$

L is, of course, considered to be in thousands of feet.

(*b*) The line shown in Fig. 31 is 6 miles long and is composed of No. 19 B. & S. gage copper wire. Under balanced bridge conditions $R_1/R_2 = 5$, $R_s = 1672$ ohms. Find X in thousands of feet if the average temperature of the copper wire is 30°C.

10. A single street car is operating on a line which is 8000 ft long. The line is energized with a 600-volt generator at one end and with a 550-volt generator at the other. The overhead trolley wire has a resistance of 0.05 ohm/1000 ft, and the bonded rail return has a resistance of 0.01 ohm/1000 ft. If the car draws 200 amp (assumed constant), find the distance between the car and the 600-volt generator when the voltage across the car, V_c, is a minimum. What is the minimum value of V_c? *Note*: Express V_c as a function of the distance X and set $dV_c/dX = 0$ to obtain the required information.

11. Solve for the branch currents I_A, I_B, and I_C in Fig. 32 by the branch-current method.

12. Solve for the currents in the two battery branches of Fig. 33, (*a*) by the mesh-current method, (*b*) by first transforming the 2-3-5-ohm delta to an equivalent wye.

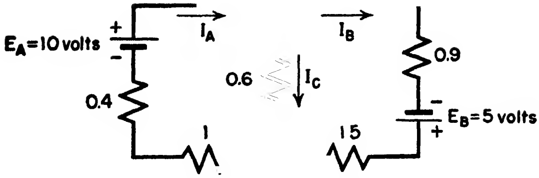


FIG. 32. See Prob. 11. (Resistance values refer to ohms.)

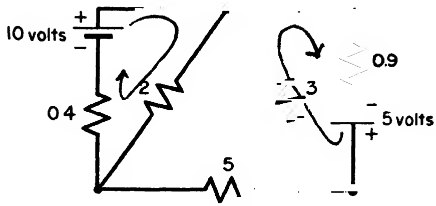


FIG. 33. See Prob. 12. (Resistance values refer to ohms.)

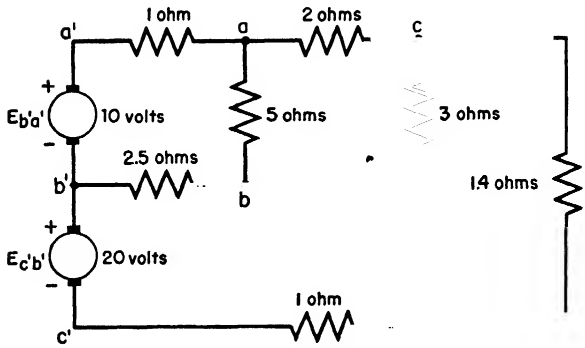


FIG. 34. See Prob. 13.

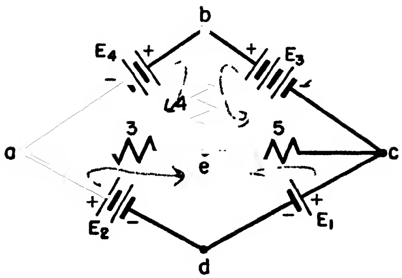


FIG. 35. See Prob. 14. (Resistance values refer to ohms.)

13. Solve for $I_{a'a}$, $I_{b'b}$, and $I_{cc'}$ in Fig. 34.

14. In Fig. 35, E_1 is a one-cell *battery*, E_2 is a two-cell battery, E_3 is a three-cell battery, and E_4 is a two-cell battery. Each cell is assumed to have an open-circuit voltage of 1.5 volts and an internal resistance of 1 ohm. Find the current through each of the batteries, and state whether the battery is delivering energy or receiving energy in each case.

15. The two d-c generators, E_1 and E_2, of Fig. 36 are operating with terminal voltages of 625 and 600 volts respectively as shown.

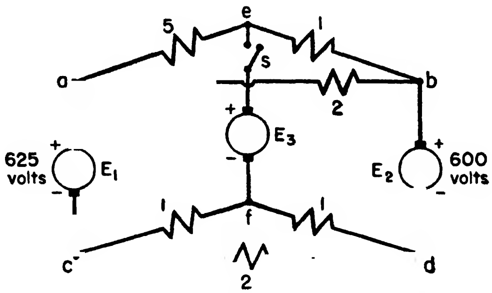


FIG. 36. See Prob. 15. (Resistance values refer to ohms.)

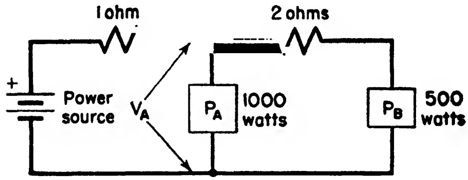


FIG. 37. See Prob. 16.

(*a*) What voltage must be developed by the E_3 generator if, when switch S is closed, zero current will flow through this generator?

(*b*) What power is dissipated in each of the 2-ohm resistors?

16. In Fig. 37, P_A and P_B are dissipative loads of 1000 and 500 watts respectively. The voltage across the P_A load is 100 volts.

(*a*) What is the power delivered by the power source?

(*b*) What is the terminal voltage of the power source?

(*c*) What current flows through the 2-ohm resistor?

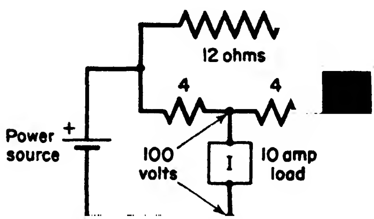


FIG. 38. See Prob. 17.

17. In Fig. 38, the 10-amp load has a voltage drop across it of 100 volts.

(*a*) What is the terminal voltage of the power source?

(*b*) What is the power delivered by the power source?

18. A particular generator has an open-circuit voltage of 0.35 volt and an internal resistance of 0.062 ohm. This generator is connected to a load through two No. 20 B. & S. gage copper wires which total 10 ft in length. What is the maximum power that can be taken by the load? Assume that the resistance of the load can be adjusted to receive maximum power. The working temperature of the copper wires is 30°C.

19. Refer to Fig. 15, page 137. $E_{b'a'}$ and $E_{c'b'}$ are d-c generators, the open-circuit voltages of each being 120 volts, and the internal resistances being each 0.2 ohm. $R_{a'a} = R_{c'c} = 1.0$ ohm and $R_{b'b} = 2.0$ ohm.

Determine the current in each of the three connecting wires and the efficiency of the three-wire transmission system (not including the losses in the generators) under the following conditions:

(a) $R_{ab} = R_{bc} = 18.8$ ohms.
(b) $R_{ab} = 12$ ohms and $R_{bc} = 50$ ohms.

In each case R_{ab} and R_{bc} represent the useful loads on the system.

20. Refer to the attenuation section (or pad as it is sometimes called) shown in Fig. 13, page 134. What is the attenuation of this section if $R_1 = 100$ ohms, $R_2 = 75$ ohms, $R_o = \sqrt{R_1R_2 + R_1^2/4}$, and $I_{out} = 0.1$ amp? Express result in decibels.

21. Design an attenuation section of the kind shown in Fig. 13, page 134, which will yield an attenuation of 20 db and have an input resistance of 500 ohms; assume that R_o of Fig. 13 is 500 ohms.

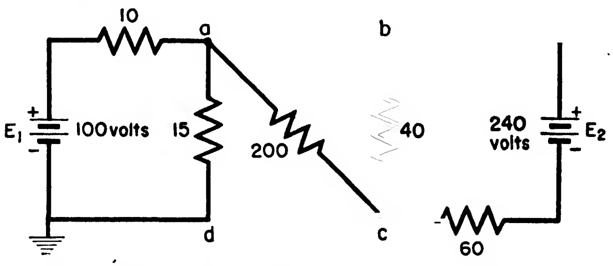


FIG. 39. See Prob. 23. (All resistance values refer to ohms.)

22. Refer to Fig. 7, page 122, and the accompanying numerical example. Find the current in the *bc* branch (in Fig. 7-a) and specify its positive direction, that is, whether $+I$ flows from *b* to *c*, or from *c* to *b*, if the 4.5-ohm resistor is replaced with a 2.5-ohm resistor and $E = 5.2$ volts.

23. Refer to the circuit arrangement shown in Fig. 39. What are the potentials of points *a*, *b*, *c*, and *d* relative to the *ground* which is indicated as being at the $-$ terminal of the E_1 battery?

24. Refer to the elementary distribution system shown in Fig. 40. Assume that the operating temperatures of all the connecting wires (assumed to be copper) are 30°C. Assume further that the lamp takes 150/110 amp even though the actual potential difference across its terminals might differ slightly from the rated value of 110 volts.

(a) What is the voltage across the lamp terminals when the motor is running under a steady load drawing 5 amp?

(*b*) What is the voltage across the lamp terminals when the motor is in the process of starting and drawing 25 amp?

25. The circuit arrangement shown in Fig. 41 is that of a potentiometer wherein R_{ab} is adjustable as indicated and can be read to five significant figures. R_{ab} may be varied between the limits of zero and R_{abc} without disturbing this latter value.

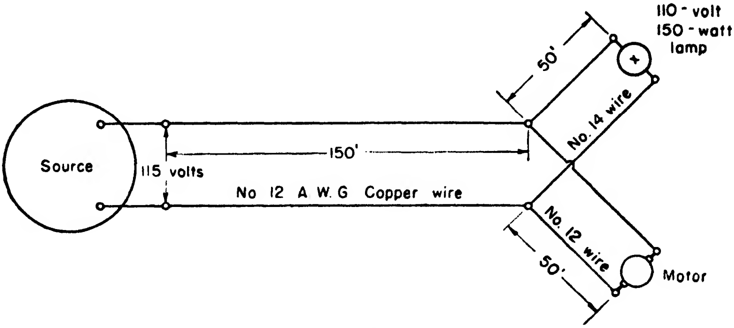


FIG. 40. See Prob. 24.

E_s is a standard cell, the voltage of which is known to five significant figures, and E_x is the unknown voltage.

(*a*) With the switch closed in the standardizing or *s* position and R_{ab} *set* to its 1018.3-ohm value, what ohmic value of R_{abc} will make $I_s = 0$? It will be observed

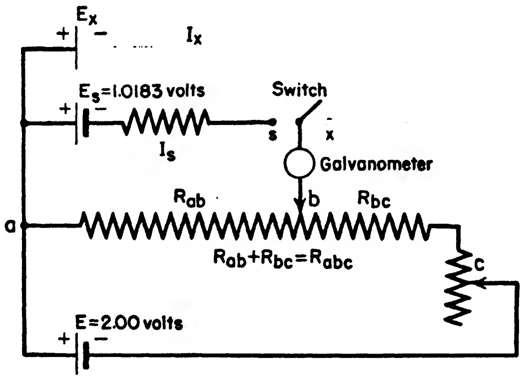


FIG. 41. The basic potentiometer circuit for measuring unknown voltages. See Prob. 25.

that R_{abc} may be adjusted at *c* without disturbing the setting of R_{ab} which has been specified.

(*b*) With R_{abc} *set* as outlined in (*a*), the switch is changed over to the *x* position and R_{ab} adjusted until $I_x = 0$. For $R_{ab} = 1550$ ohms, $I_x = 0$, find the value of E_x.

26. In Fig. 42, find the potential of point *x* relative to the ground bus, preferably by the nodal method of analysis.

27. (*a*) How many voltage equations are required to effect a network solution of the arrangement shown in Fig. 43 if the mesh-current method of analysis is employed?

(*b*) How many current equations are required to effect a network solution of the arrangement if the voltage sources are first transformed to equivalent current sources?

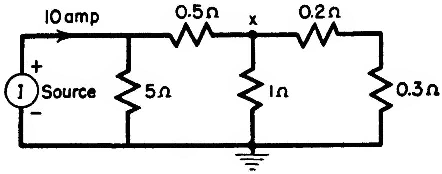


FIG. 42. See Prob. 26.

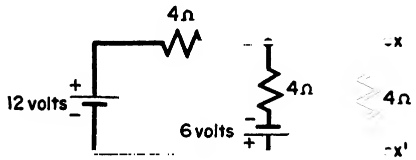


FIG. 43. See Probs. 27, 28, 31, 32, and 33.

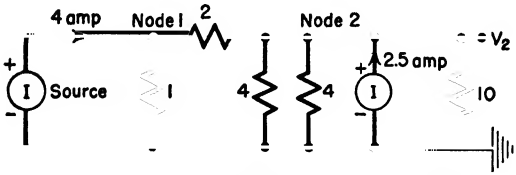


FIG. 44. See Prob. 29. (Resistance values refer to ohms.)

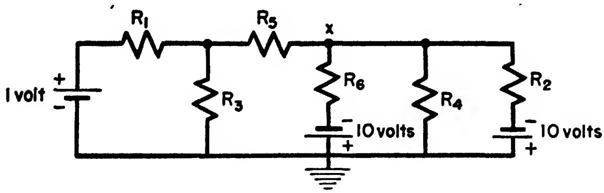


FIG. 45. See Prob. 30.

28. Transform the specified voltage sources in Fig. 43 to equivalent current sources and find the voltage of point x relative to point x' by the nodal method of analysis.

29. Find V_2 relative to ground in Fig. 44.

30. Find the voltage of point x relative to ground in the network shown in Fig. 45. $R_1 = R_2 = R_3 = R_4 = R_5 = R_6 = 1$ ohm.

31. In Fig. 43, find the open-circuit voltage across the terminals x and x' by considering the voltage sources one at a time and combining the results in accordance with the principle of superposition.

32. What is the Thévenin-equivalent resistance, R_t, looking back into the network shown in Fig. 43 from terminals x and x'?

33. In Fig. 43, a voltage source of 4 volts having an internal resistance of 1 ohm is placed across terminals x and x' with the negative terminal at x. Find the current which flows through this 4-volt source.

34. Can the battery and galvanometer of a conventional Wheatstone bridge (Fig. 30) be interchanged and still have the four resistance arms function as a Wheatstone bridge?

35. Refer to Fig. 46. Show by means of mesh-current solutions of the network that I_3 with E in the 1 position is precisely the same as I_1 if E is changed to the 3 position.

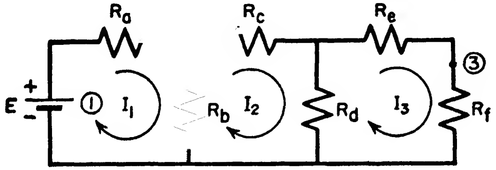


FIG. 46. See Probs. 35, 36, and 37.

Note: Employ determinants and the conventional mesh-current solution notation, namely, R_{11}, R_{12}, R_{13}, R_{21}, R_{22}, and so on, in order that the problem may be solved concisely for any numerical values of R_a, R_b, R_c, R_d, R_e, and R_f.

36. In Fig. 46 it will be assumed that $E = 11$ volts and that

$$R_a = R_c = R_e = 2 \quad \text{ohms}$$

$$R_b = R_d = R_f = 4 \quad \text{ohms}$$

(*a*) If the $R_e - R_f$ branch is broken at point 3, what is the open-circuit voltage which appears across this break?

(*b*) What is the resistance looking back into the network from the break in the $R_e - R_f$ branch?

(*c*) What current flows in the $R_e - R_f$ branch when it is closed as shown in Fig. 46?

37. In Fig. 46 it will be assumed that $E = 12$ volts and that

$$R_a = R_a = R_e = 2 \quad \text{ohms}$$

$$R_b = R_d = R_f = 4 \quad \text{ohms}$$

(*a*) Find the current in the R_c branch by opening this branch and using Thévenin's theorem.

(*b*) Check the above result by solving for I_2 and employ the conventional mesh- or loop-current method of solution.

38. In Fig. 47, it will be observed that every branch of the network has been traversed by either loop current I_1 or loop current I_2.

(*a*) Can a correct physical solution be obtained from the following loop-current. equations?

$$(R_1 + R_2 + R_3 + R_6)I_1 + (R_2 + R_6)I_2 = E_1 - E_3$$

$$(R_2 + R_6)I_1 + (R_2 + R_4 + R_5 + R_6)I_2 = 0$$

Explain the physical inconsistency involved in this type of solution.

(*b*) What is the correct number of loop-current equations required in Fig. 47?

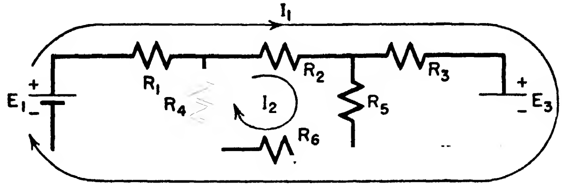


FIG. 47. See Prob. 38.

(*c*) Might a physically correct solution be obtained in Fig. 47 if, to the two loop currents already shown (I_1 and I_2), another loop current is added which traverses the E_3, R_3, R_5 loop?

39. In Fig. 48 is shown a decade voltage attenuator of the type used in some oscillators. Although in practice the voltage e is an a-c voltage, the present problem may be worked on a d-c basis. R_3, in Fig. 48, is known to be 10 ohms.

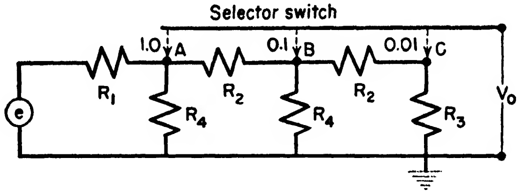


FIG. 48. See Probs. 39, 40, and 41.

(*a*) Find the value of R_2 if, with point B 0.1 volt above ground, V_o (the output voltage) is 0.01 volt with the selector switch set at C.

(*b*) Find the value of R_4 required to make the output voltage, V_o (with the selector switch at B) equal to 0.1 volt if the voltage at point A is maintained at 1 volt above ground.

40. What value of R_1 should be employed in Fig. 48 in order that the resistance looking back into the network from the V_o terminals will be the same with the selector switch set at either A, B, or C? It is known that $R_2 = 90$ ohms, $R_3 = 10$ ohms, and $R_4 = 11.1$ ohms.

41. The general requirements of the decade attenuator shown in Fig. 48 are

(*a*) $V_C = 0.1V_B$ and $V_B = 0.1V_A$.

(*b*) The resistance looking back into the network from the V_o terminals be the same with the selector switch set at either the A, B, or C positions.

Show that these requirements are met when

$$(1)\ R_2 = 9R_3 \qquad (2)\ R_1 = 10R_3 \qquad (3)\ R_4 = \tfrac{10}{9} R_3$$

CHAPTER VII

The Electric Field

1. General Concept. Any region in which a stationary electric test charge Q_t would experience a force, if placed there, is *by definition* an electric field. Thus any region in the vicinity of charged bodies is an electric field.

The region between positively and negatively charged areas in the atmosphere may, for example, be so electrified that cloud-to-cloud

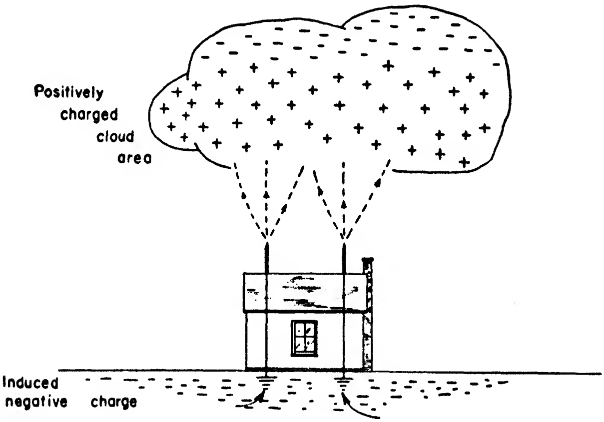


FIG. 1. Illustrating lightning-rod operation for the case of a positively charged cloud directly over the tips of the rod.

lightning discharges occur. Or the region between a positively charged cloud area and the house shown in Fig. 1 might become so electrified that electrons are attracted from the earth (by the + cloud) to the tips of the lightning rods where they discharge into the atmosphere. In so doing, the electrons tend to neutralize the + cloud and prevent flashover. The converse happens when a negative cloud area forms over the house; the rods become positively charged, and negative charges from the cloud are attracted to ground by way of the rods.

The electric field which is present in an electrical device sometimes determines to a great extent the physical operation of the device. An

example of this is shown in Fig. 2. The electric forces developed on the electrons e (emanating from cathode k) by the positively charged plate p and the negatively charged grid g determine the rate at which electrons travel from cathode to plate. Since this rate of travel defines $+i$ in the output circuit of this simple amplifier, the output current depends for its value upon the instantaneous electric field which exists within the vacuum tube.

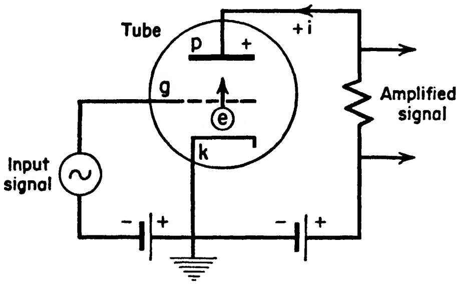


FIG. 2. Amplifier operation determined by electric field within the vacuum tube.

Quantitatively, the strength of an electric field is measured in terms of the $\mathcal{E}$ vector.

2. The $\mathcal{E}$ Vector. The electric intensity vector $\mathcal{E}$ has already been employed to some extent in connection with the brief explanation of electrical conduction given in Chapter IV. In Section 2 of that chapter, there was introduced the definition

$$\mathcal{E} = \frac{dV}{dl}\bigg]_{\text{max.}} = \frac{\mathbf{f}}{Q} \tag{1}$$

In this connection, it will be remembered that V was reckoned as potential *drop*.

Equation (1) serves to define the $\mathcal{E}$ vector[1] as directed precisely along the same path as the $\mathbf{f}$ vector since Q (a scalar quantity) is taken to be a positive charge in this definition. From the definition contained in equation (1), it is plain that we may consider the $\mathcal{E}$ vector quantitatively either as (1) the maximum space rate of change of electric potential drop, $\frac{dV}{dl}\Big]_{\text{max.}}$ and measure it in volts per unit length, or as (2) the *force per unit charge* and measure it in, say, newtons per coulomb.

The usefulness of the $\mathcal{E}$ vector in electric field theory is plainly evident when we consider that the potential *drop* between two points in the electric field (for example, points a and b) may be calculated from the line integral

$$V_{ab} = \int_a^b \mathcal{E} \cdot d\mathbf{l} \tag{2}$$

[1] The $\mathcal{E}$ vector may be pronounced as would the **E** vector because the word *vector* will distinguish it from the scalar potential E. In more advanced texts, it is not uncommon to find the symbol E representing both *potential* and the negative of the *potential gradient* where the context of the subject matter is such as to permit this dual usage.

or the voltage rise between these two points may be calculated from

$$E_{ab} = \int_a^b - \mathcal{E} \cdot d\mathbf{l} \tag{2-a}$$

Also in determining forces on electric charges which are present in electric fields we may write, from equation (1),

$$\mathbf{f} = \mathcal{E}Q \tag{3}$$

where, if Q is positive, the direction of $\mathbf{f}$ is the same as the direction of the $\mathcal{E}$ vector, and if Q is negative the direction of $\mathbf{f}$ is in the $-\mathcal{E}$ direction. This latter concept has already been invoked in explaining metallic conduction on a qualitative basis in Chapter IV.

Before equations (2) or (3) can be used to advantage, however, ways and means must be found of evaluating $\mathcal{E}$ from the physical data that are usually available.

3. An Analogy to $\mathcal{E} = \left.\dfrac{dV}{d\mathbf{l}}\right]_{\text{max.}} = -\left.\dfrac{dE}{d\mathbf{l}}\right]_{\text{max.}}$. The fact that the electric field may be described either in terms of *potential* or in terms of *potential gradient* is somewhat analogous to the fact that a plot of ground can be described either in terms of *elevation* or in terms of *elevation gradients* (or simply grades). In one case the field is described in terms of a scalar function (potential or elevation), and in the other case in terms of gradients, that is, space rates of change of the scalar quantities. These space rates of change are vector quantities since they have directions as well as magnitudes.

Reference to the contour map of any plot of ground will show that along lines of equal elevation (contour lines), the space rate of change of elevation is zero, whereas in the direction normal to contours the space rate of change of elevation is maximum. If h represents height, say above some arbitrary reference level, then $\left.\dfrac{dh}{d\mathbf{l}}\right]_{\text{max.}}$ is directed normal to the contour lines and is the value normally referred to unless $dh/d\mathbf{l}'$ along some other path $\mathbf{l}'$ is specified, in which case the magnitude of $dh/d\mathbf{l}'$ is less than $\left.\dfrac{dh}{d\mathbf{l}}\right]_{\text{max.}}$ because the latter designation implies that we shall choose $\mathbf{l}$ to be in such a direction as to obtain the maximum space rate of change of elevation.

Where the civil engineer usually defines gradients (or simply grades) as increases in elevation per unit of horizontal distance, the electrical engineer defines potential gradients as increases in electric potential per

unit distance. Where the civil engineer uses contour lines to depict differences in elevation, the electrical engineer uses equipotential surfaces (or lines) to depict differences in electrical potential.

4. Equipotential Surfaces. As the name implies, an equipotential surface is a surface all points of which are at the same potential. Equipotential surfaces were employed in Chapter IV as a guide in calculating the resistance of irregular conductors. There the potential differences were established by RI drops in the conducting material. In this chapter we are more concerned with equipotential surfaces and equipotential lines in the vicinity of charged bodies where little or no current flows. In the present case, the significant electrical effects are not confined to well-defined paths or circuits, and concepts other than those incorporated in circuit theory must be employed. Field maps are a valuable aid in the study of electric fields.

An important concept in connection with field mapping is that all points on the surface of a conductor which possesses a static charge are at the same potential. This follows from the fact that *if* potential differences existed *on* or *within* the conductor, the charge would redistribute itself until all points reached the same potential. The fact that "surplus" or "free" charge resides on the surface is a natural consequence of the mutual forces of repulsion which the individual surplus charges exert on one another. That the interior of the conductor is at the same potential as its surface follows from the same argument. In distributing itself on the surface of the conductor, the surplus charge so arranges itself that the force on a hypothetical test charge placed within the conductor is acted on by forces (in all directions), the resultant of which is *zero*. It follows that the magnitude of the $\mathcal{E}$ vector is zero within a conductor which possesses static charge.[2]

From the definition of the $\mathcal{E}$ vector which has been given in equation (1), it is not difficult to visualize an $\mathcal{E}$ vector as originating at a positively charged conductor and terminating at a negatively charged conductor since the $\mathcal{E}$ vectors are zero within the conductors and are defined in direction by the path which a positive test charge would follow if it were free to move in the region between the two conductors.

If surface a (shown in cross-section in Fig. 3) is a conductor surface, the $\mathcal{E}$ vector originates at that surface if surface a is of higher potential than surface b. That the $\mathcal{E}$ vector is directed normal to the conductor surface follows from the fact that the force on a test charge has zero

[2] In the normal current-carrying conductors which we have previously considered only a relatively small component of the total $\mathcal{E}$ vector is required to produce the desired current flow. Where good conductors are used, as in transmission lines, this component is often negligibly small in comparison to the component of $\mathcal{E}$ which exists in the insulating material between the two line conductors.

component directed *along* the equipotential surface. In a similar manner we find that the $\mathcal{E}$ vector shown in Fig. 3 is directed normal to surface b because the $\mathbf{f}$ in equation (1) cannot by definition possess a component which is directed along an equipotential surface.

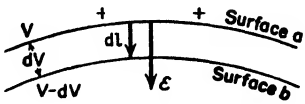


FIG. 3. The $\mathcal{E}$ vector crosses equipotential surfaces at right angles.

Where a hollow conductor (or even an open grill work like a screen) completely surrounds a region and this conductor is held at ground potential, the enclosed region is electrically shielded from the effects of all exterior $\mathcal{E}$ vectors. A *striking* example of the shielding properties of a metal enclosure is shown in Fig. 4 where the artificial lightning is seen flashing over the front tire to ground.

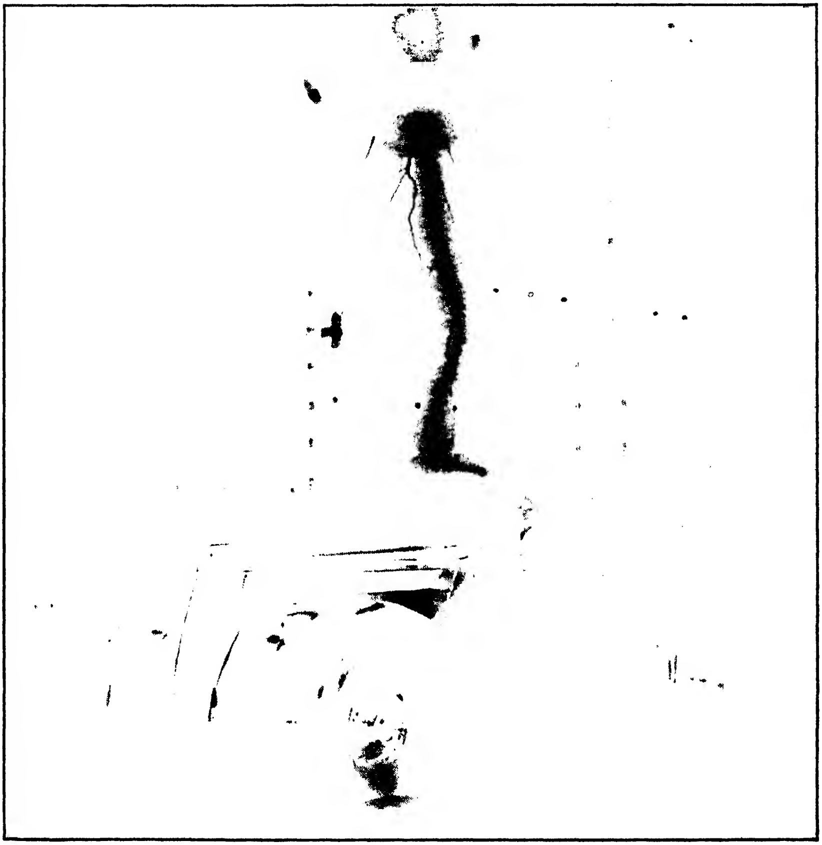


FIG. 4. Illustrating the shielding properties of a metal enclosure. (Courtesy of Westinghouse Electric Corp.)

Example. In Fig. 5 are shown the cross-sections of several equipotential surfaces which are present in the region between two equally and oppositely charged conductors. (The method employed in obtaining a map of this kind is discussed in the following section.)

If the two charged conductors are the only charges in the vicinity, line C is the cross-section of a *zero* potential surface since *no work* is required to move an exploring or test charge along this line however far extended. Since the conductors are assumed to be equally and oppositely charged, the *resultant force* acting on our test charge when placed at any point along the line C *is normal* to the path of motion (line C) and hence

$$V = \frac{\int f \cos 90° \, dl}{Q_t} = 0$$

where it is understood that the dl's are directed along line C.

If line A in Fig. 5 is the cross-section of a surface which is 200 volts above zero, then to move a positive test charge of 1 coulomb from point M to point

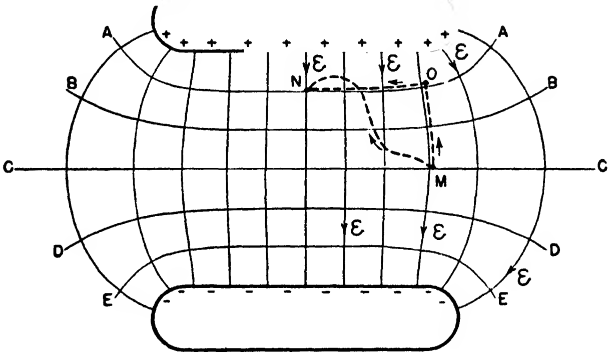


FIG. 5. Equipotential lines A, B, C, D, and E in the region between two equally and oppositely charge conductors.

N requires 200 joules of work ($W = EQ$), and this amount of work is required regardless of the path traversed.

5. Evaluation of $\mathcal{E}$ from a Field Map. Although more precise methods of evaluating $\mathcal{E}$ will be given later, the graphical method employed here has the advantage of not being clouded with mathematical details. The electric field map shown in Fig. 5 is constructed in the same general manner as that outlined on page 88; the two guiding principles being:

(1) Flow lines (which in this case are indicated by the $\mathcal{E}$ vectors) cross equipotential lines (which are cross-sections of equipotential surfaces) at right angles.

(2) The ratio of the *flow* dimension (of each curvilinear square) to the *equipotential* dimension of the square is unity.

In Fig. 5, the charged conductors are indicated in cross-section by the heavy lines, and it is assumed that each of these equally and oppositely charged conductors has a dimension normal to the page which is relatively large compared to the cross-sectional dimensions shown. The cross-sectional view which is shown is near the center of the long dimension, and the medium intervening between the two conductors may be any insulating material or even evacuated space.

Before the drawing of an electric field map, as in Fig. 5, is started, it is known that the $\mathcal{E}$ lines emanate at the + charges and terminate at the − charges as indicated. Then, treating the conductor cross-sections as equipotential lines, the draftsman visualizes the approximate location of another equipotential line not far from the conductors as, for example, line A or E. If line A (or E) is not correctly placed it will be impossible to construct a set of curvilinear squares which meet the two specified requirements. In connection with requirement (2) listed above, it should be noted that maps may be obtained by employing ratios other than unity but, where the drafting process is one of trial and success as it is here and where the use of a pair of dividers is a valuable aid in maintaining a fixed ratio, unity ratio is most often selected.

There is only *one correct* map and it is obtained graphically once the entire area is divided into curvilinear squares which meet the two basic requirements. Some practice is, of course, required before the draftsman can hope to produce a good map, particularly in regions of irregular geometrical configuration. Requirement 2 in these cases demands somewhat more careful study than is necessary for our purposes since only regular geometries will be encountered in this text.

It will be assumed that the reader could after a few trials construct the map shown in Fig. 5 and interpret it in light of equation (1). Our immediate concern is the evaluation of the $\mathcal{E}$ vector from a field map of the kind shown in Fig. 5, and it should be realized that the incremental distances between the equipotential lines correspond to the dl's of equations (2) and (2-a). In a good map these incremental distances are sufficiently short so that, as a good approximation, the magnitude of $\mathcal{E}$ is essentially constant over the distance $\Delta \mathbf{l}$. As applied to a field map, equation (2) reads

$$\Delta V \text{ (between potential lines)} = \mathcal{E} \cdot \Delta \mathbf{l} \tag{4}$$

and since ΔV and $\Delta \mathbf{l}$ are usually known quantities the map provides a method of evaluating $\mathcal{E}$ as well as giving a potential contour map of the region in which we are interested.

Example. *Interpretation of Fig.* 5. If the two conductors shown in Fig. 5 are maintained at a potential difference of 600 volts and potentials are reck-

oned relative to the potential of line C, the following potential distribution is obtained:

Upper conductor	+300 volts	Lower conductor	−300 volts
Potential line A	+200 volts	Potential line E	−200 volts
Potential line B	+100 volts	Potential line D	−100 volts

This potential distribution is obtained from the fact that the map in this case divides the known potential difference (600 volts) into 6 equal incremental potential differences (ΔV's) each of 100 volts.

The magnitude of the $\mathcal{E}$ vector is essentially constant over the entire mid-section because in this region the Δl's are essentially of the same length. Since $\mathcal{E} = \Delta V/\Delta l$, from equation (4), it follows that the magnitude of the vector $\mathcal{E}$ varies inversely as the separation of the equipotential lines on the field map. For example, the distance of separation between the B and C lines at the extreme edge of the mapped area where *fringing* is quite pronounced is very nearly twice the mid-section separation between these lines. Hence the magnitude of the $\mathcal{E}$ vector is about twice as large at the mid-section of the map as it is at the extreme edge of the mapped area shown in Fig. 5.

If it were known that the mid-section separation of the conductors is 0.06 m, the mid-section value of Δl would be 0.01 m since there are 6 equal incremental lengths along the center vertical line between conductors. Hence,

$$\text{mid-section value of } \mathcal{E} = \frac{100}{0.01} = 10{,}000 \quad \text{volts/m}$$

$$\text{outer edge value of } \mathcal{E} = \frac{100}{0.02} = 5000 \quad \text{volts/m (approx.)}$$

The magnitude of the force developed on an electronic charge (-1.6×10^{-19} coulomb) would, in accordance with equation (3), be

$$\text{mid-section value of } f = 10{,}000 \times 1.6 \times 10^{-19} = 1.6 \times 10^{-15} \text{ newton}$$

$$\text{outer edge value of } f = 5000 \times 1.6 \times 10^{-19} = 0.8 \times 10^{-15} \text{ newton}$$

Equally important is the direction of the force, and this is clearly indicated by the direction of the $\mathcal{E}$ vectors. In this case since a negative charge is involved the direction of the force is in the $-\mathcal{E}$ direction that is in the − to + direction.

6. Electric Flux Density, D. An auxiliary physical quantity which is useful in describing electric fields in isotropic[3] media is *electric flux*

[3] "Having the same properties in all directions." Quartz crystals, for example, are non-isotropic. Boundary layers between different metals and metal oxides are in general non-isotropic, and the classical field theory as it is presented here cannot be applied to the boundaries. Some boundary layer phenomena which are important to the electrical engineer will be considered in a later chapter.

density. Electric flux density is *defined* as

$$\mathbf{D} = \epsilon_0\epsilon_r\mathcal{E} = \epsilon\mathcal{E} \quad \text{(units of flux/unit area)} \tag{5}$$

where ϵ_0 is the permittivity of free space and is equal to $\dfrac{1}{36\pi \times 10^9}$ in rationalized mks units

ϵ_r is the relative permittivity of the medium, assumed constant

$\epsilon = \epsilon_0\epsilon_r$ for convenience in writing.

Thus **D** is *defined* as a space-distributed vector quantity that is linearly related to the $\mathcal{E}$ vector which has previously been defined.

Where an $\mathcal{E}$ field exists, a **D** field also exists, and vice versa. The immediate advantage to be gained by the use of **D** is that it can be evaluated easily from a theorem given in a following section in certain cases where the direct evaluation of $\mathcal{E}$ is rather awkward and involved. After **D** has been found, $\mathcal{E}$ follows directly from equation (5). But aside from this immediate use, **D** has long been accepted as a useful concept and was first introduced by Maxwell as a useful working concept in his classical treatment of electromagnetic wave propagation.

If at this stage the reader is curious about the units which have been assigned to **D** in equation (5), he should examine the definition of **D** from a dimensional point of view. Dimensionally,

$$\mathcal{E} = V^1l^{-1} = W^1Q^{-1}l^{-1} = Q^{-1}f^1 \quad \text{(from basic definitions)}$$

$$\epsilon = Q^2f^{-1}l^{-2} \quad \text{(from Coulomb's inverse square law)}$$

Hence

$$D = \epsilon^1\mathcal{E}^1 = Q^1l^{-2}$$

which implies that D is dimensionally an area density of some kind of *flux.* This flux is dimensionally the same as electric charge Q; and it is precisely this fact that makes **D** a very useful concept even though it is a mathematically formulated concept.

Example. Consider a small spherically distributed charge of $+4 \times 10^{-6}$ coulomb to be the only charge in the vicinity. See Fig. 6.

Let it be required to find $\mathbf{D} = \epsilon\mathcal{E}$ at a point which is 0.1 m from the center of the charge by first evaluating $\mathcal{E}_x = -\left.\dfrac{dE}{dx}\right]_{\text{max.}}$.

A general expression for E as a function of x may be obtained readily in this case as

$$E_x = \int_x^\infty f'\,dx$$

where f' is the force of repulsion per unit positive test charge which would be exerted on a test charge which is x distance from the center of the parent ($+4 \times 10^{-6}$ coulomb) charge.

$$E_x = \int_x^\infty \frac{Q}{4\pi\epsilon_0 x^2}\,dx = (9 \times 10^9)(4 \times 10^{-6}) \int_x^\infty \frac{dx}{x^2}$$

$$E_x = 36{,}000 \left[-\frac{1}{x} \right]_x^\infty = \frac{36{,}000}{x} \quad \text{volts} \quad (\text{if } x \text{ is in meters})$$

E_x tells us the absolute potential of any point along the x axis which is outside the spherically distributed charge, and it is plain from the symmetry of the

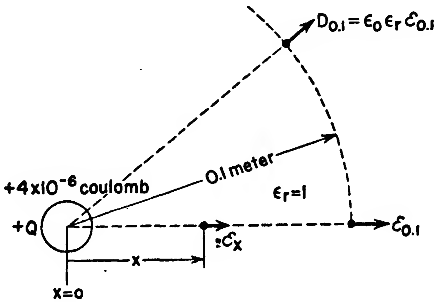


FIG. 6. Illustrating $\mathbf{D} = \epsilon_0 \epsilon_r \boldsymbol{\mathcal{E}}$.

charge that the x axis might have been chosen along any radial line emanating from the center of this charge with the same results.

$$\mathcal{E}_x = -\frac{dE_x}{dx} = -\left[-\frac{36{,}000}{x^2} \right] = \frac{36{,}000}{x^2}$$

At a distance of 0.1 m from the charge center, $x = 0.1$ m and

$$\mathcal{E}_{0.1} = 3.6 \times 10^6 \quad \text{volts/m} \quad (\text{directed as shown in Fig. 6})$$

$$D_{0.1} = \frac{1}{36\pi \times 10^9} \times 3.6 \times 10^6 = 3.18 \times 10^{-5} \quad \text{coulombs/sq m}$$

The present example shows only how **D** might be evaluated in terms of $\boldsymbol{\mathcal{E}}$. The converse evaluation, the more important one, will be considered after Gauss' theorem has been discussed. See Section 8.

7. Electric Flux, ψ. The concept of *flux* is quite widely used as, for example, luminous or light flux, magnetic flux, and in the present instance electric flux. Flux is a scalar quantity which is obtained by forming the product of a vector area and the *component* of some distributed space vector which is *normal* to the face of the area; that is,

$$\text{flux} = \text{area} \times \text{flux density} \tag{6}$$

provided the flux density employed in this connection is normal to the face of the area as shown in Fig. 7.

Electric flux is usually represented by the symbol ψ, and in general the flux crossing an area A is

$$\psi = (D \cos \theta]_{\mathbf{A}}^{\mathbf{D}})A \tag{7}$$

where ψ is in *coulombs of flux* if D is expressed in coulombs per square meter and A in square meters.

It will be observed that the right-hand member of equation (7) is simply the dot product[4] of vectors **D** and **A** if by definition we agree

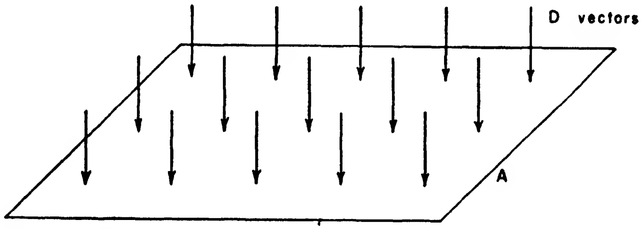


FIG. 7. Illustrating **D** vectors at right angles to the face of area A.

that the direction of a vector area is to be at right angles to the face of the area, as shown in Fig. 8. This is the most concise way of describing the *direction* of a vector area, that is, assign by definition the direction of an area as being normal to the face of the area. In fact this is the only direction which can be associated with a vector area to define uniquely its directional properties. (A little thought on the matter will convince the reader that the physical orientation of an area in space, as, for example, the sail area of a boat, is an important factor in many physical problems.)

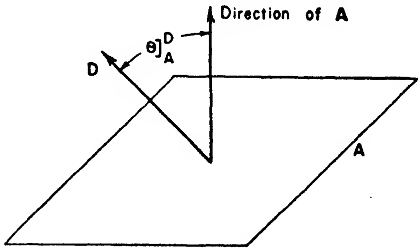


FIG. 8. The direction of **A** is normal to the face of A.

If both **D** and **A** are considered as vector quantities, equation (7) takes the simple form of

$$\psi = \mathbf{D} \cdot \mathbf{A} \tag{7-a}$$

[4] The dot product has been considered in connection with the product of force and distance in defining work or energy. Thus energy which is a scalar quantity is obtained by the dot product of two vector quantities. See page 15.

where it is understood that the right-hand member must be actually evaluated as indicated by the operations shown in equation (7).

Although electric flux has the dimensions of electric charge and, in the rationalized mks system, the *primary unit of flux is the coulomb*, electric flux is usually symbolized as ψ rather than as Q in order to call attention to the fact that the *projected* effect of the parent charge (and not the parent charge itself) is under discussion.

In thinking about flux crossing or piercing the face of an area we must of course think of "something" crossing the area. In the case of electric flux, this "something" may be considered to be *projected charge*. This concept is tangible and at the same time useful in many types of problems. If, for example, we treat ψ as *projected charge* and accept it as such, any time rate of change of ψ is immediately associated with a current ($I_d = d\psi/dt$ coulombs/sec or amp). Thus we can conceive very logically of a current flowing in free space which is absolutely devoid of charge carriers. This type of current is encountered in condensers, in wave guides, around radio antennas, and in fact wherever a time rate of change of electric flux is present. $I_d = d\psi/dt$ is called *displacement current* to distinguish it from ordinary conduction current where real charge carriers transport the charge. Maxwell introduced the concept of displacement current as a mathematically formulated concept in his prophecy of electromagnetic wave propagation many years before this type of wave propagation had been demonstrated experimentally. For the next few sections in this text, however, we shall not be concerned with displacement current because stationary (or electrostatic) flux only is under discussion.

Example 1. Let it be required to find the flux which crosses a spherical area which encloses and is concentric to the 4×10^{-6} coulomb charge shown in Fig. 6; it is assumed that the radius of the enclosing spherically shaped area is 0.1 m.

It has been shown in the preceding example that

$$D_{0.1} = 3.18 \times 10^{-5} = \frac{10^{-4}}{\pi} \quad \text{coulombs/sq m}$$

Since $\mathbf{D}$ is everywhere directed normally to the face of the area, $\theta]_{\mathbf{A}}^{\mathbf{D}}$ equals zero, and in this case

$$\psi = DA = \frac{10^{-4}}{\pi} \times 4\pi 0.1^2 = 4 \times 10^{-6} \quad \text{coulomb}$$

As a matter of fact it can be shown that the flux crossing a spherical enclosing area of any radius will be 4×10^{-6} coulomb in this case.

Example 2. Let it be required to find the electric flux which pierces the $a \times b$ area of the dielectric material shown in Fig. 9 (neglecting the non-

uniform distribution and curvature of the $\mathcal{E}$ and hence the **D** vectors around the periphery).

The known data are

$$d = 0.01 \text{ m} \quad a = 0.2 \text{ m} \quad b = 0.4 \text{ m} \quad \epsilon_r \text{ of dielectric} = 5$$

The potential difference between the plates (that is, between $x = d$ and $x = 0$ in Fig. 9) is maintained at 300 volts.

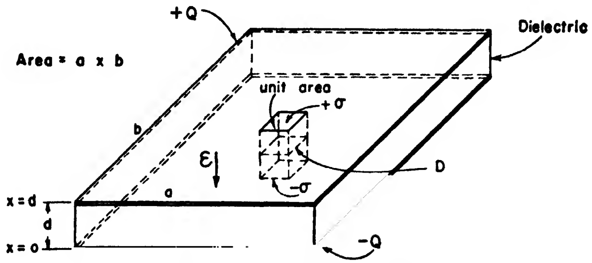


FIG. 9. Two parallel conducting plates separated by a dielectric.

All other designations on this figure (which, incidentally, represents a parallel-plate condenser) are for use in connection with capacitance calculations, considered later.

Since the $\mathcal{E}$ vectors (except near the edges) are straight parallel lines, they have constant magnitude over the distance from $x = d$ to $x = 0$. This magnitude is simply evaluated as

$$\mathcal{E} = \frac{\Delta V}{\Delta l} = \frac{300}{0.01} = 30{,}000 \text{ volts/m} \quad \text{(directed as shown)}$$

$$\mathbf{D} = \epsilon_0 \epsilon_r \mathcal{E} = \frac{1}{36\pi \times 10^9} \times 5 \times (3 \times 10^4) = 1.327 \times 10^{-6} \quad \text{coulomb/sq m.}$$

Since these **D** vectors cross normal to the face of the dielectric (the $a \times b$ plane),

$$\psi = (1.327 \times 10^{-6}) \times 0.08 = 1.06 \times 10^{-7} \quad \text{coulomb}$$

It is this value of electric flux which will later be shown to give rise to *current through the condenser* provided $d\psi/dt$ has a finite value.

8. Gauss' Theorem. A theorem due to Gauss states that (in rationalized mks units): "the *net electric flux* directed outwardly through any surface which encloses a net or surplus charge of Q coulombs *is Q coulombs of flux*." The theorem applies to any charge configuration or distribution and to any shape of enclosing surface. If the net charge enclosed is negative, the flux is directed inwardly across the face of the enclosing surface.

The general thought behind this theorem is shown schematically and in cross-section in Fig. 10, where each of the lines shown may be thought

of as a *tube* of electric flux. Since each positive coulomb of surplus charge, regardless of its configuration, establishes in the surrounding space 1 coulomb of electric flux the enclosing area 1 shown (in cross-section) in Fig. 10 will be pierced by 3 coulombs of electric flux if each of the + signs refers to 1 coulomb of surplus charge. Since each coulomb of negative charge is a sink or termination area for 1 coulomb of electric flux, it follows that area 2 in Fig. 10 is pierced by only 1 coulomb of electric flux since the enclosed area is made large enough to provide within its interior a termination area for two (of the three) coulombs of electric flux which emanate from the positively charged body. Where area 2 is considered as the enclosing area, however, the net or surplus

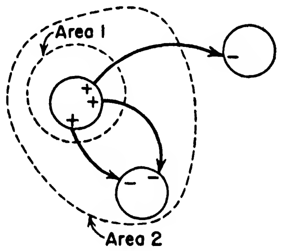


FIG. 10. Two Gaussian surfaces in cross-section.

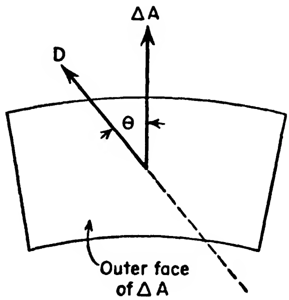


FIG. 11. A patch ΔA of an enclosing or Gaussian surface.

charge enclosed is only 1 positive coulomb, and this net charge is shown as giving rise to 1 coulomb of electric flux which crosses outwardly through the enclosing area.

To consider this important theorem in more minute detail we might take a small patch of the enclosing area, say $\Delta\mathbf{A}$ in Fig. 11 and investigate what happens at the surface of this incremental area. For an arbitrary charge distribution within the enclosure (not all of which is shown in Fig. 11) and for an arbitrary shape of enclosing surface, it is plain that $\mathbf{D}$ will in general be directed at some angle other than 90° to the face of $\Delta\mathbf{A}$, that is, other than zero degrees from the direction of $\Delta\mathbf{A}$. Let the angle between the direction of $\Delta\mathbf{A}$ and the direction of $\mathbf{D}_A$ be θ as shown in Fig. 11. If $\Delta\mathbf{A}$ is sufficiently small, $\mathbf{D}_A$ is essentially constant over this entire area, and the flux crossing $\Delta\mathbf{A}$ is

$$\Delta\psi = (D_A \cos\theta)\,\Delta\mathbf{A} \quad \text{coulombs of flux}$$

since the other right-angled component of $\mathbf{D}_A$ (namely, $D_A \sin\theta$) is in the plane of $\Delta\mathbf{A}$ and hence does not cross $\Delta\mathbf{A}$.

If, now, the entire enclosing area is considered piece by piece as it was

above, the total flux crossing the enclosing surface is

$$\Sigma(D_A \cos \theta)\ \Delta A = \psi \tag{8}$$

and this number of coulombs of flux must originate at Q coulombs of charge within the enclosing area.

Example 1. In the first example of the preceding section a charge of 4×10^{-6} coulomb was enclosed within a spherically shaped enclosing surface, the radius of which was taken to be 0.1 m.

By means other than Gauss' theorem, it was shown that the electric flux crossing this enclosing surface was 4×10^{-6} coulomb.

Employing Gauss' theorem we might have arrived at this same conclusion without calculation and in addition appreciated the statement made in that example to the effect that the radius selected for the enclosing surface would not affect the amount of flux crossing that surface.

Example 2. We shall assume here that the known data in Fig. 9 are

$d = 0.01$ m $\quad a = 0.20$ m $\quad b = 0.40$ m $\quad \epsilon_r$ of dielectric $= 5$

σ (surface charge per square centimeter of upper plate)

$$= +1.327 \times 10^{-10} \quad \text{coulomb/sq cm}$$

σ (surface charge per square centimeter of lower plate)

$$= -1.327 \times 10^{-10} \quad \text{coulomb/sq cm}$$

The non-uniform distribution of this surface charge which is actually present near the periphery of the plates will be neglected and a uniform charge distribution over the lower surface of the upper plate and the upper surface of the lower plate will be assumed.

If, now, we pass an enclosing surface completely around the upper plate and include in this enclosing surface an $a \times b$ plane of the dielectric, we know from Gauss' theorem that the flux crossing the enclosing surface is

$$\psi = (1.327 \times 10^{-10}) \times 20 \times 40 = 1.06 \times 10^{-7} \quad \text{coulomb} \quad \text{(of flux)}$$

since the total charge on the upper plate is $Q = \sigma A$, where, for the uniform surface charge distribution assumed, A is the inside or lower flat surface of the upper plate.

Under the conditions assumed, all the flux crossing the enclosing surface which has been envisioned will cross an $a \times b$ plane of the dielectric; that is, 1.06×10^{-7} coulomb of flux originating at the upper plate and the same amount terminating at the lower plate. Hence the flux density in the dielectric is

$$\mathbf{D} = \frac{1.06 \times 10^{-7}}{0.2 \times 0.4} = 1.327 \times 10^{-6} \quad \text{coulomb/sq m}$$

$$\mathcal{E} = \frac{D}{\epsilon_0 \epsilon_r} = \frac{1.327 \times 10^{-6}}{\dfrac{1}{36\pi \times 10^9} \times 5} = 30{,}000 \quad \text{volts/m}$$

$$V_{\text{between plates}} = 30{,}000 \times 0.01 = 300 \text{ volts potential difference}$$

This application of Gauss' theorem shows how a potential difference (300 volts in this case) may be evaluated from an assigned distribution of charge on the conductors whose potential difference is being evaluated. The present example reverses the procedure employed in Example 2 of the preceding section where the potential difference was specified.

9. Forces Developed on Charged Surfaces in an Electric Field. The device used here to illustrate forces which are developed on charged surfaces is shown in Fig. 12. In order to obtain a uniform flux distribution

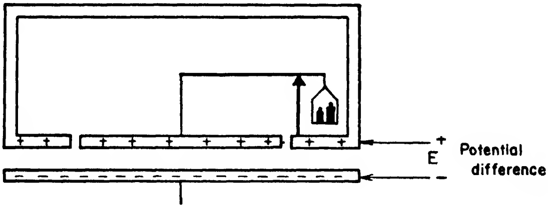


FIG. 12. An electrometer used in measuring forces developed on a charged surface.

in the air gap between the movable plate and the lower plate shown in Fig. 12 we surround the movable plate with a guard ring, the details of which are shown in Fig. 13.

If we assign $+\sigma$ units of charge to each unit of area of the upper plate and $-\sigma$ units of charge to each unit of area of that portion of the

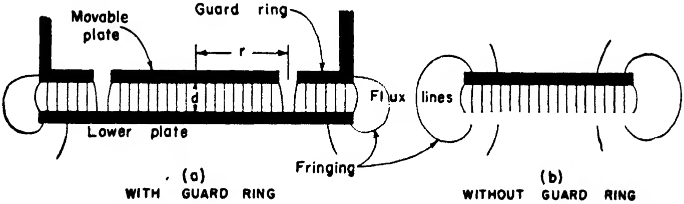


FIG. 13. Flux distributions with and without guard ring.

lower plate which is directly below the movable plate we find by the application of Gauss' theorem that

$$D = \sigma \quad \text{(in the intervening air gap)}$$

$$\mathcal{E} = \frac{D}{\epsilon} = \frac{\sigma}{\epsilon} \quad \text{(in the gap)} \tag{9}$$

We know from equation (1) that the force which would be developed on a unit test charge placed in the electric field which is present *between* the two plates is equal to $\mathcal{E}$, or σ/ϵ units of force. This, however, is not the force exerted per unit of charge on the upper plate because the σ/ϵ force

on a unit charge in the gap is the result of two equal components of force, one attributable to the charge on the upper plate and one attributable to the charge on the lower plate. The force developed *on* the distributed charge of the upper plate is due only to the lower plate's contribution, namely, $\sigma/2\epsilon$ units of force per unit charge on the upper plate. Thus,

$$f' \text{(force developed } on \text{ each unit of surface charge)} = \frac{\sigma}{2\epsilon} \tag{10}$$

in the parallel-plate arrangement shown in Fig. 12.

The total force on the movable plate is

$$f_p = f'Q = f'\sigma A = \frac{\sigma^2 A}{2\epsilon} \tag{11}$$

which may be rearranged algebraically to show that

$$f_p = \frac{\sigma^2 A^2}{2\epsilon A} = \frac{Q^2}{2\epsilon A} \tag{12}$$

where Q is the total charge on the movable plate. $\epsilon = \epsilon_0 \epsilon_r = \epsilon_0$ since the arrangement shown requires that the intervening medium between the plates be air. ($\epsilon_0 = \dfrac{1}{36\pi \times 10^9}$ in rationalized mks units.)

A is the area of the upper plate which, if circular, would be πr^2. The slit between the movable plate and the guard ring shown in Fig. 13 is presumably very small, and by measuring r out to the middle of this small slit as indicated we can essentially compensate for the small percentage of the total force which is developed along the vertical sides of the movable plate. It will be observed in Fig. 13 that no flux crosses from the movable plate to the guard ring since these objects are at the same potential.

Example. *An Absolute Coulombmeter and Voltmeter.* A rearrangement of equation (12) will show that

$$Q = \sqrt{2\epsilon_0 A f_p} \tag{13}$$

The physical arrangement shown in Fig. 12 might therefore be used to measure Q in terms of the mechanical quantities A and f.

The potential difference between the plates is

$$E = \mathcal{E} d = \frac{\sigma}{\epsilon_0} d = \sqrt{\frac{2f_p d^2}{\epsilon_0 A}} \tag{14}$$

since it is known from equation (11) that $\sigma = \sqrt{2\epsilon_0 f_p / A}$. The arrangement shown in Fig. 12 may therefore be used to measure potential difference in terms of A and f and hence function as a voltmeter. This method of measuring

potential difference is not used except in certain educational laboratories where the demonstration of the principle of forces on charged surfaces is of more importance than the voltage measurement itself.

10. Energy Stored in an Electric Field. If we assume that the plates shown in Fig. 12 are energized as indicated and think of moving the upper plate an incremental distance (Δl) upward, we find that the work done is

$$\Delta W = f_p\,\Delta l = \frac{\sigma^2 A}{2\epsilon}\,\Delta l \quad \text{[See equation (11).]} \tag{15}$$

provided the plate is moved only an infinitesimally short distance (so that f_p remains essentially constant) or else E is increased in order to hold f_p constant during the period of the movement of the plate.

The work expended by the operator is given in equation (15) and appears in the form of energy stored in the added increment of electric field, namely, the incremental volume $A\,\Delta l$. The stored energy per unit volume is then

$$w = \frac{W}{A\,\Delta l} = \frac{\sigma^2}{2\epsilon} \tag{16}$$

or since $\sigma = \epsilon\mathcal{E}$,

$$w = \frac{\epsilon\mathcal{E}^2}{2} \quad \text{(joules/cu m, in mks units)} \tag{16-a}$$

where $\mathcal{E}$ is expressed in volts per meter and $\epsilon = \epsilon_0\epsilon_r$.

Example. Consider two rectangular metal plates (10 m $\times$ 0.1 m) separated by a dielectric material which has an $\epsilon_r = \pi$ and a thickness of 0.0004 m (or 0.4 mm).

Let it be required to find the total energy stored in the electric field which exists between the two metal plates if these plates are maintained at a potential difference of 2400 volts.

$$\mathcal{E} = \frac{2400}{0.0004} = 6 \times 10^6 \quad \text{volts/m}$$

$$W = w \times \text{(volume)} = \frac{\epsilon\mathcal{E}^2}{2} \times 0.0004 = \frac{\pi}{36\pi \times 10^9} \times \frac{36 \times 10^{12}}{2} \times 0.0004$$

or

$$W = 0.2 \quad \text{joule}$$

This energy may be stored in the electric field simply by momentarily subjecting the plates to the 2400-volt potential difference. If a good grade of dielectric is employed this energy remains stored for a long time (several days at least) and may be released at any time during this period (say by a screw driver having an insulated handle), and the stored energy will manifest itself

in an electrical discharge which flashes and sounds like a miniature discharge of lightning.

11. Breakdown or Dielectric Strength of Insulating Materials. The magnitude of $\mathcal{E}$ at which an insulating material "breaks down," that is, ceases to function as an insulator, is called the *dielectric strength* of the material, and is frequently tabulated in handbooks in the secondary unit, *kilovolts per millimeter.* The dielectric strengths of some of the more common insulators are shown in Table I together with the relative permittivity (or dielectric constant) and resistivity where the latter values are well defined. The dielectric strength of a material varies widely with the conditions under which the tests are performed, that is, with the shape of the electrodes employed, the moisture content, the temperature, and so on. Tests are usually carried out on samples which are about 1 or 2 mm in thickness; and the results of these tests must be applied with caution to thicknesses which are more than about ten times the thickness of the test sample.

The dielectric strength of a material is not related in any definite or known manner to the dielectric constant (or relative permittivity, ϵ_r) of the material as may be seen from Table I.

TABLE I

Material	Dielectric Strength (kv/mm)	Dielectric Constant ϵ_r	Resistivity (ohms/cm cube) (approx.)
Air (atm. pr., 0°C)	3	1.00058	
Bakelite, Micarta-213	up to 31.4	5	5×10^{11}
Glass (ordinary)	8 to 9	5 to 9	9×10^{13}
Mica	21 to 28	5 to 7	up to 2×10^{17}
Oil (transformer)	10 to 35	2.5	
Paper (impregnated)	12 to 20	3 to 4	5×10^{14}
Rubber (hard)	70	2 to 3.5	1×10^{18}
Wood (maple-paraffined)	4.6	4.1	3×10^{10}

If the $\mathcal{E}$ vector becomes sufficiently great in magnitude in an insulating material, the forces developed on the otherwise bound charges within the molecular structures of the material will be sufficiently great to overcome the internal molecular binding forces. The internal structure of dielectric molecules is such that under the influence of an $\mathcal{E}$ field the center of positive charge (within a molecule or group of molecules) is not coincident with the center of negative charge. This is shown schematically in Fig. 14 where each + and − combination indicates a molecular structure containing both positive and negative charges which are capable of being shifted slightly within the confines of the molecular

structure when subjected to an $\mathcal{E}$ field. The forces on these molecular structures (called dipoles) will under normal potential gradients be insufficiently large to produce any free charge carriers. At the breakdown point, however, these dipoles are ruptured and stripped of their internal charges. The free charges which are thus produced change the erstwhile insulator into a conductor of sorts. Breakdown usually damages the insulator permanently because the heat generated by the conduction current which follows an internal breakdown chars or otherwise deteriorates the material.

The insulation material of electrical apparatus is designed with factors of safety which range from about 3 to 10; owing principally to the fact

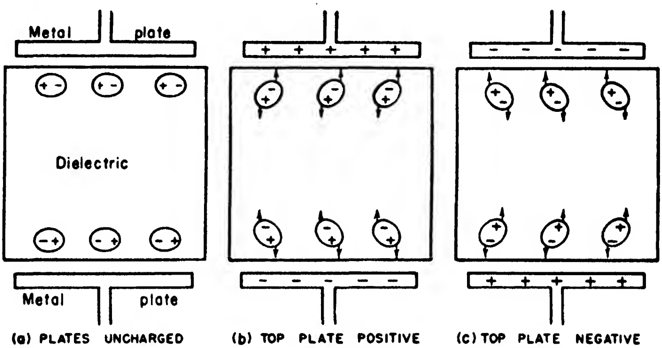


FIG. 14. Bound molecular charges experience electric forces.

that the insulation materials are not strictly uniform in their dielectric strengths.

Example. One specification or requirement sometimes imposed on insulation material is that the material shall have a dielectric strength such that it can withstand twice rated voltage plus 1000 volts. By rated voltage is meant the voltage to which the apparatus is normally subjected in actual operation.

Let it be required to apply this specification to the dielectric material of the preceding example where the insulation material is subjected to a potential gradient of 6×10^6 volts/m at the assumed rated voltage of 2400 volts. (See example, Section 10.)

The paper, which is 0.4 mm thick, is assumed to have a dielectric strength of 14 kv/mm.

This paper can withstand a voltage of

$$14 \times 0.4 = 5.6 \text{ kv} \quad \text{or} \quad 5600 \text{ volts}$$

$$\text{Twice-rated voltage} + 1000 \text{ volts} = 4800 + 1000 = 5800 \text{ volts}$$

Hence this particular paper fails to meet the specification in this instance by a 200-volt margin.

12. Changes in the Electric Field as It Crosses Boundaries. The $\mathcal{E}$ vector in general suffers a change in magnitude and direction when it passes the boundary between two dielectric media having different dielectric constants. That this is true may be seen by applying the fundamental principles which have already been considered.

First, consider the case in which the $\mathcal{E}$ vectors are known to cross the boundary (between the two dielectrics) at right angles as shown in Fig. 15. In this case the magnitude of the $\mathcal{E}$ vector changes when it crosses

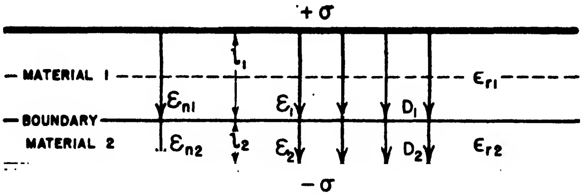


FIG. 15. $\mathcal{E}_{n1}/\mathcal{E}_{n2} = \mathcal{E}_1/\mathcal{E}_2 = \epsilon_{r2}/\epsilon_{r1}$ since **D** is continuous across boundary here.

the boundary but the direction of the vector remains unchanged. This follows from the fact that the **D** vectors remain unchanged since all the electric flux emanating from the positively charged upper plate shown in Fig. 15 is known to terminate at the negatively charged plate directly

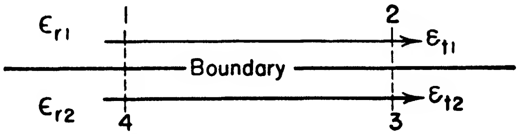


FIG. 16. Tangential components of $\mathcal{E}$ are equal. $\mathcal{E}_{t1} = \mathcal{E}_{t2}$.

below. Since the flux and, hence, **D** are continuous, $\mathcal{E} = \mathbf{D}/\epsilon_0\epsilon_r$ must change if ϵ_r changes in magnitude while the continuous **D** vector crosses the boundary from the ϵ_{r1} region into the ϵ_{r2} region. Thus,

$$\frac{\mathcal{E}_1}{\mathcal{E}_2} = \frac{\mathcal{E}_{n1}}{\mathcal{E}_{n2}} = \frac{\epsilon_{r2}}{\epsilon_{r1}} \quad (\mathcal{E} \text{ normal to boundary}) \tag{17}$$

where the subscripts n refer to normal components of the $\mathcal{E}$ vectors.

Where the $\mathcal{E}$ vectors are tangential to the boundary as shown in Fig. 16, it follows that they must be equal in magnitude since the voltage drop from points 1 to 2 in this figure must be the same as the voltage drop from points 4 to 3 if the actual $\mathcal{E}$ field is tangent to the boundary. This follows from an application of Kirchhoff's emf law around the 1234 loop shown in Fig. 16. If we assume that the distances 23 and 41 in this loop

are infinitesimally small, Kirchhoff's voltage law reads

$$V_{12} + V_{34} = \mathcal{E}_{t1}l_{12} + \mathcal{E}_{t2}l_{34} = 0$$

and since l_{34} is the negative of l_{12}

$$\mathcal{E}_{t1} = \mathcal{E}_{t2} \quad (\mathcal{E} \text{ tangential to boundary}) \tag{18}$$

If an $\mathcal{E}_1$ vector meets a boundary between two dielectrics at some angle θ_1 as shown in Fig. 17, it will in general suffer a change in both magnitude and direction. The $\mathcal{E}_1$ vector, upon entering material 2 from material 1, will take on a new value, say $\mathcal{E}_2$ as shown in Fig. 17 if $\epsilon_{r2} > \epsilon_{r1}$. The relationship between θ_1 and θ_2 may be established in terms of the

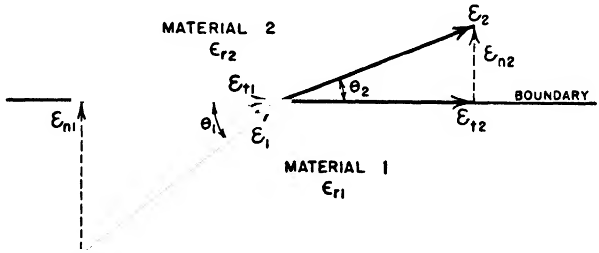


FIG. 17. Illustrating $\mathcal{E}_{n1}/\mathcal{E}_{n2} = \epsilon_{r2}/\epsilon_{r1}$, $\mathcal{E}_{t1}/\mathcal{E}_{t2} = 1$, and $(\tan\theta_1)/(\tan\theta_2) = \epsilon_{r2}/\epsilon_{r1}$.

two principles just developed for the normal and tangential components of the $\mathcal{E}$ vector. By breaking both $\mathcal{E}_1$ and $\mathcal{E}_2$ into normal and tangential components as shown in Fig. 17, we find

$$\frac{\mathcal{E}_{n1}}{\mathcal{E}_{n2}} = \frac{\epsilon_{r2}}{\epsilon_{r1}} = \frac{\mathcal{E}_1 \sin\theta_1}{\mathcal{E}_2 \sin\theta_2} \tag{19}$$

by the normal component principle stated in equation (17). Also

$$\frac{\mathcal{E}_{t1}}{\mathcal{E}_{t2}} = 1 = \frac{\mathcal{E}_1 \cos\theta_1}{\mathcal{E}_2 \cos\theta_2} \tag{20}$$

if we consider the tangential components in light of equation (18).

Combining equations (19) and (20) we have

$$\frac{\tan\theta_1}{\tan\theta_2} = \frac{\epsilon_{r2}}{\epsilon_{r1}} \tag{21}$$

If, for example, $\epsilon_{r2} = 2\,\epsilon_{r1}$ and θ_1 is known to be 45°, then

$$\tan\theta_2 = \tan\theta_1 \frac{\epsilon_{r1}}{\epsilon_{r2}} = 1 \times 0.5 \quad \text{or} \quad \theta_2 = 26.6°$$

From the development preceding equation (21), it is plain that this equation accounts only for the change in ϵ_r at the boundary and does

not apply to non-isotropic media. Neither does it apply to boundaries where charge distributions are present which act as sources or sinks of electric flux, because in these cases the **D** vectors change in magnitude as they cross the boundary. Boundaries of this kind will be encountered when we consider batteries, thermocouples, and the like in Chapter XIV.

13. Gauss' Theorem Applied to Cylindrical Charge Distributions. Charge distributions which are essentially cylindrical in shape are frequently encountered in practice; particularly in transmission line work and in vacuum tubes having electrodes which are coaxial cylinders. It is usually of importance to be able to determine $\mathcal{E}$ at any point in the immediate vicinity of these charge distributions. If the conductors are very long relative to the distance of separation, as is usually the case in

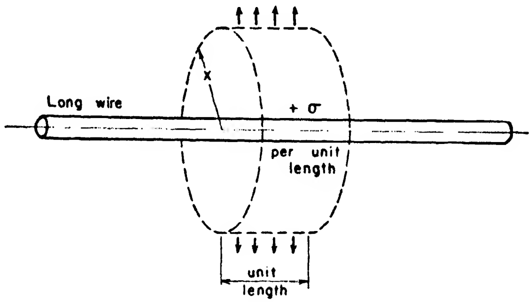


FIG. 18. An enclosing (or Gaussian) surface about a uniformly distributed line charge.

transmission lines, the electric field established by a single cylindrically distributed charge is such that the $\mathcal{E}$ vectors are directed radially outward from a positively charged wire (or radially *toward* a negatively charged wire).

The problems usually encountered in practice can be attacked by *assigning* a surface charge density of $+\sigma$ units of charge per unit length to one of the conductors and $-\sigma$ units of charge per unit length to the other conductor. The actual magnitude of these charge densities is usually immaterial but the process of assigning these charge densities places the conductors in the same energized state as would the application of a potential difference to the conductors. It turns out that the potential difference developed between the conductors (as a result of *assigning* σ units of charge to the conductors) is directly proportional to the magnitude of σ.

The result of assigning $+\sigma$ units of charge per unit length to a cylindrical conductor is shown in Fig. 18. A cylindrically shaped enclosing (or Gaussian) surface of the type shown in Fig. 18 will be crossed or pierced

at only the $2\pi x(1)$ surface since the ends of this right cylinder are parallel to the flux lines and hence not pierced by the flux lines. From Gauss' theorem, we can write directly

$$D_x = \frac{\sigma}{2\pi x(1)} = \frac{\sigma}{2\pi x} \tag{22}$$

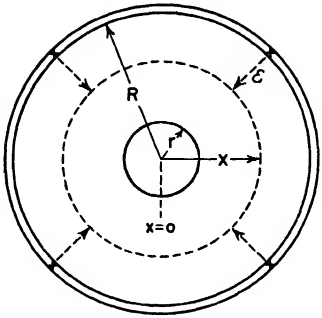


FIG. 19. Cross-sectional view of long coaxial conductors. R is inside radius of outer conductor; r is the radius of inner conductor.

where D_x is the flux density, say in coulombs per square meter, at any radial distance x from the axis of the charged conductor; and the associated $\mathcal{E}$ vector at this point is

$$\mathcal{E}_x = \frac{\sigma}{\epsilon 2\pi x} \quad \text{volts/m} \tag{23}$$

ϵ has its usual meaning, $\epsilon_0\epsilon_r$, and $\epsilon_0 = 1/(36\pi \times 10^9)$ in the units employed in equation (23).

As applied to the coaxial cylinders shown in Fig. 19, we note that

$$\mathcal{E}_x = \frac{\sigma}{\epsilon 2\pi x} \quad \text{volts/m directed radially inward} \tag{23-a}$$

If r is the radius of the inner conductor (and R the inside radius of the outer conductor as shown), the potential difference between the conductors, written as a function of σ, is

$$V = \int_R^r \mathcal{E}(-dx) = \frac{\sigma}{\epsilon 2\pi}\int_r^R \frac{dx}{x} = \frac{(18 \times 10^9)\sigma}{\epsilon_r} \ln\frac{R}{r} \tag{24}$$

The minus sign in the first integral accounts for the fact that we are proceeding in the $-x$ direction as we sum up $\mathcal{E}\, dl$ from $x = R$ to $x = r$. It follows that

$$\frac{V}{\sigma} = \frac{18 \times 10^9}{\epsilon_r} \ln\frac{R}{r} \quad \text{volts/coulomb/m} \tag{25}$$

It will be observed that the right member of the above equation is a function only of the ratio of the dimensions and the ϵ_r of the insulating material between the two conductors. Although equation (25) will be of some immediate use in the problems of the present chapter, it will take on added importance when the subject of capacitance is discussed in Chapter XIII.

A cross-sectional view of the two conductors of a parallel-wire line is shown in Fig. 20. The conductors have dimensions normal to the page which are relatively great compared to D, the center-to-center separation

of the line wires. Let it be required to find the potential difference between the two line wires as a function of the magnitude of the arbitrarily assigned charge density σ. This problem differs from the coaxial problem just considered in that the charged conductors taken individually produce $\mathcal{E}$ vectors which are coincident in direction *only* along the line joining the axes of the two conductors; and, along this line, the magnitudes of these two individual components of the resultant $\mathcal{E}$ are equal only at the point midway between the two conductors.

We may proceed to write an expression for the resultant $\mathcal{E}$ along the line joining the conductor centers and find $V_{ab} = \int_r^{D-r} \mathcal{E}_{\text{res.}}\, dx$; or we may calculate first the voltage drop from a to b in Fig. 20 due to the $+\sigma$ line charge, then the voltage drop from a to b due to the $-\sigma$ line charge, and finally apply the principle of superposition to find the resultant voltage from a to b. Since the latter method is capable of extension to

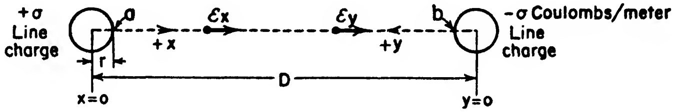


FIG. 20. Cross-sectional view of parallel-wire transmission line. D is the center-to-center separation; r is the radius of each conductor.

more than two line charges (and in fact will be used later where four line charges are involved) it will be applied to the two-wire case in order to illustrate the procedure. (The same principle might have been applied to the coaxial problem presented above, but this is given as a student exercise. See Prob. 25.)

Considering first the $+\sigma$ line charge only,

$$V_{ab} \text{ (due to } +\sigma) = \int_r^{D-r} \mathcal{E}_x\, dx = \int_r^{D-r} \frac{\sigma}{\epsilon 2\pi x}\, dx$$

$$= \frac{18 \times 10^9 \sigma}{\epsilon_r} \int_r^{D-r} \frac{dx}{x}$$

$$= \frac{18 \times 10^9 \sigma}{\epsilon_r} \ln \frac{D-r}{r} \quad \text{volts} \tag{26}$$

Shifting our attention now to the $y = 0$ origin in Fig. 20 and considering only the $-\sigma$ line charge,

$$V_{ab} \text{ (due to } -\sigma) = \int_{D-r}^{r} \mathcal{E}_y\, (-dy) = \int_r^{D-r} \frac{\sigma}{\epsilon 2\pi y}\, dy$$

$$= \frac{18 \times 10^9 \sigma}{\epsilon_r} \ln \frac{D-r}{r} \quad \text{volts} \tag{27}$$

The minus sign used in connection with dy in the first integral accounts for the fact we are integrating in the $-y$ direction as we proceed from point a (which is $D - r$ distance from $y = 0$) to point b (which is r distance from the origin).

By combining the component voltages calculated in equations (26) and (27), the actual voltage that exists between the conductors is obtained:

$$V_{ab} = \frac{36 \times 10^9 \sigma}{\epsilon_r} \ln \frac{D - r}{r} \quad \text{volts} \tag{28}$$

Again it is seen that the ratio V_{ab}/σ is a function only of the ratio of physical dimensions $D - r$ and r and the relative permittivity of the medium intervening between the two line wires.

Equation (28) applies to those cases where the centers of the line charges are essentially coincident with the axes of the conductors because we have taken our origins at the conductor centers. If the distance of separation D is about twenty times the radius of a conductor r, equation (28) is essentially correct. With D less than about $10r$, the attraction between the $+\sigma$ line charge and the $-\sigma$ line charge shifts these two line charges toward one another somewhat, and to obtain correct results in this case we shift the origins used in Fig. 20 toward one another slightly (but still well within the conductors). The detailed manner in which this proximity effect is accounted for is considered later.

The electric field map of the region in the vicinity of two equally and oppositely charged cylindrical conductors is shown in Fig. 21. The lines joining the conductors are $\mathcal{E}$ lines, and the circles enclosing the conductors are equipotential lines, the straight vertical line being the true or absolute zero potential line.

Example. Let it be required to find the force experienced by an electronic charge (-1.6×10^{-19} coulomb) and the associated acceleration when the electron finds itself in the electric field established by the two charged cylinders shown in Fig. 19. The known data are

$$\epsilon_r = 1 \quad R = 2.4 \text{ cm} \quad r = 0.5 \text{ cm}$$

potential difference between conductors = 300 volts

Making use of equations (23-a) and (25) we find that, for x measured from the center of the inner conductor,

$$\mathcal{E}_x = \frac{V}{x \ln \dfrac{R}{r}} = \frac{300}{x \ln \dfrac{2.4}{0.5}} = \frac{300}{1.57x} \quad \text{newtons/coulomb}$$

It will be remembered from equation (1) that newtons per coulomb is as much a unit of $\mathcal{E}$ as is volts per meter which we have been using in the last few sections where voltage calculations were involved. Hence

$$f = \mathcal{E}Q = \frac{(300)(1.6\times10^{-19})}{1.57x} = \frac{3.06 \times 10^{-17}}{x} \quad \text{newton}$$

Since this force is plainly directed toward the outside conductor the sign of Q in the above expression is not considered, because in general it is easier to

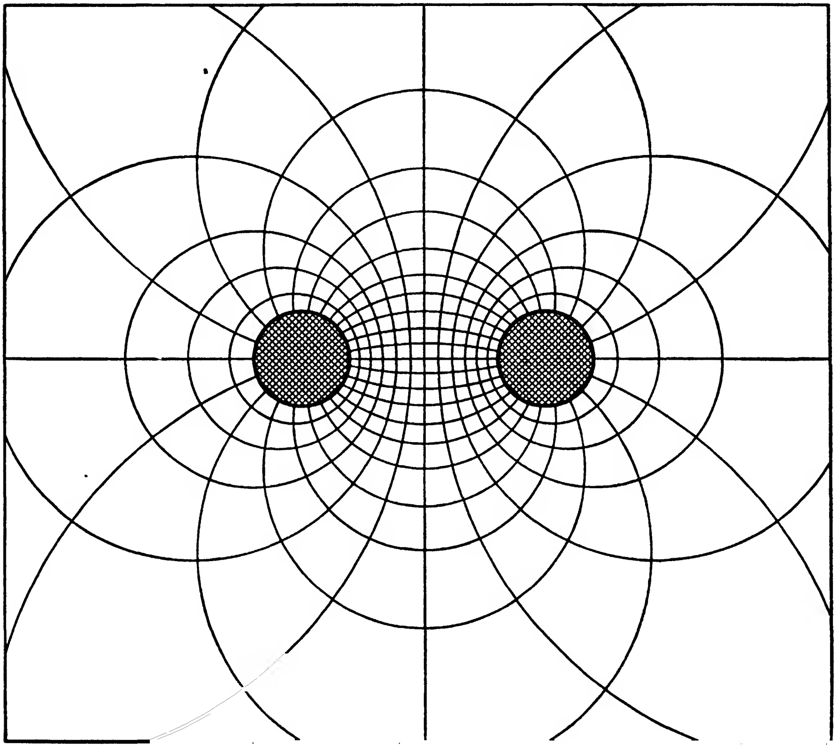


FIG. 21. Electric field map of the region near two long parallel conductors.

remember that $+Q$ goes in the $+\mathcal{E}$ direction and that $-Q$ goes in the $-\mathcal{E}$ direction (except in generators) than it is to account for these simple facts in the equations.

Since the mass of an electron is 9.1×10^{-31} kg,

$$\text{acceleration } a = \frac{f}{m} = \frac{3.06 \times 10^{-17}}{9.1 \times 10^{-31}x} = \frac{3.36 \times 10^{13}}{x} \quad \text{m/sec}$$

With this acceleration, it is apparent that the electron (if not otherwise restrained) will stay in this field only a relatively short period of time. The

problem of determining the time of flight, say from the outer surface of the inner conductor to the inner surface of the outer conductor, is interesting but not particularly simple. See Prob. 31.

PROBLEMS

1. A charge of 2×10^{-5} coulomb is located in a region where the $\mathcal{E}$ vector has a magnitude of 500 volts/cm. Express the magnitude of the force experienced by this charge in newtons and in pounds. $\left[\frac{\text{(No. of) lb}}{\text{newton}} = 0.2247.\right]$

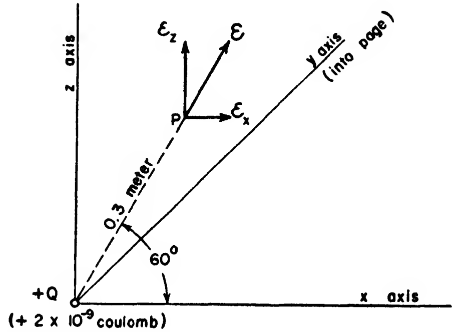


FIG. 22. Illustrating x and z components of the vector $\mathcal{E}$. Point P is in the xz plane and $y = 0$.

2. In Fig. 22 is shown a $+2 \times 10^{-9}$ coulomb charge at the origin of a set of right-angled axes. What are the magnitudes of the resultant $\mathcal{E}$ vector, the x-axis component $\mathcal{E}_x$, and the z-axis component $\mathcal{E}_z$ at point P? A free-space medium is assumed.

3. Assume that a charge of 6×10^{-9} coulomb is uniformly distributed over the surface of a metal sphere having a radius of 1 cm. Find the potential difference between the surface of the sphere and a point which is 10 cm from the center of the sphere. $\epsilon_r = 1$.

4. Consider two parallel metal plates of relatively large area separated in air by a distance of 0.002 m as shown in Fig. 23. A hypothetical test charge of 10^{-10} coulomb placed between the plates (and not too near an outer boundary) experiences a force of 0.000005 newton directed from plate B to plate A.

Which plate is at the higher potential, and by how many volts?

5. Refer to Fig. 24. The charge shown is assumed to be uniformly distributed over the surface of the sphere.

(a) Calculate the values of both E and $\mathcal{E}$ at $l = 1$, 2, and 3 m and compare your results with those shown on the graph.

(b) What is the area under the $\mathcal{E}$ versus l curve between $l = 2$ and $l = 3$ m expressed in volts?

6. Refer to Fig. 23. Assuming that the plates are maintained at a potential difference of 2000 volts find the magnitude and direction of the $\mathcal{E}$ vector in the electric field between the plates.

7. Refer to Fig. 5, page 166.

(*a*) Find the force which an electron will experience at point N if the charged conductors are maintained at a potential difference of 1200 volts and the distance of separation of these conductors along the mid-section where point N is located is 2 cm.

(*b*) What would be the approximate value of the force experienced by an electron in its travel along one of the two outermost flow lines shown in Fig. 5 as the electron crossed the CC equipotential line?

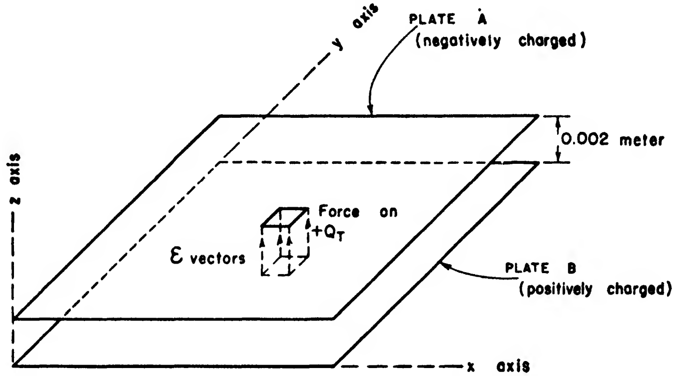


FIG. 23. Two parallel conducting plates equally and oppositely charged. See Probs. 4, 6, 19, and 22.

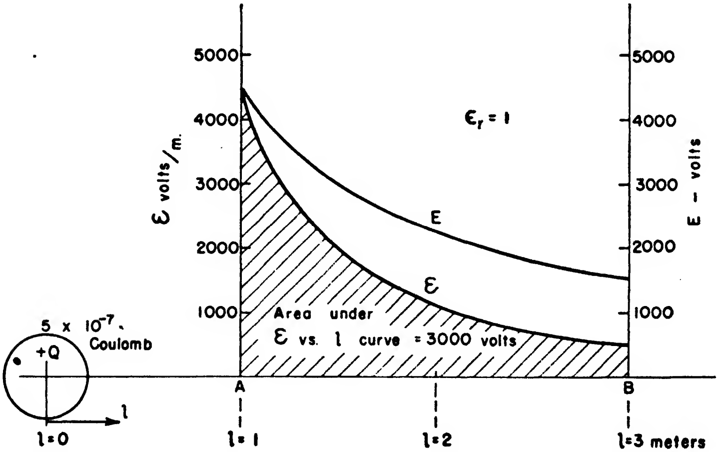


FIG. 24. Illustrating that area under an $\mathcal{E}$ versus l curve equals voltage. See Probs. 5 and 11.

8. (*a*) Add one more flow line (or $\mathcal{E}$ line) to the outer edge of Fig. 5, page 166, by graphically extending the equipotential lines in the directions dictated by the principles of curvilinear field mapping.

(*b*) From the graph obtained in (*a*), determine the approximate value of the $\mathcal{E}$ vector at the point where the outermost flow line crosses the CC equipotential line,

assuming that the conductors are maintained at a potential difference of 1000 volts and that the mid-section separation of these conductors is 2 cm.

Note: Since the outermost curvilinear square which will be obtained between the B and C equipotential lines is relatively large it should be subdivided into at least four parts before part (b) is answered. Wherever the opposite sides of a curvilinear square turn out to be distinctly different in size and contour, the square should be subdivided in accordance with the principles of mapping fields to see that it is truly a curvilinear square. If the square in question will not subdivide indefinitely (that is, into 4, then 16, and so on, smaller squares), the supposed curvilinear square is in error and adjustments must be made in either the equipotential lines or the flow lines or both until this type of subdivision is possible.

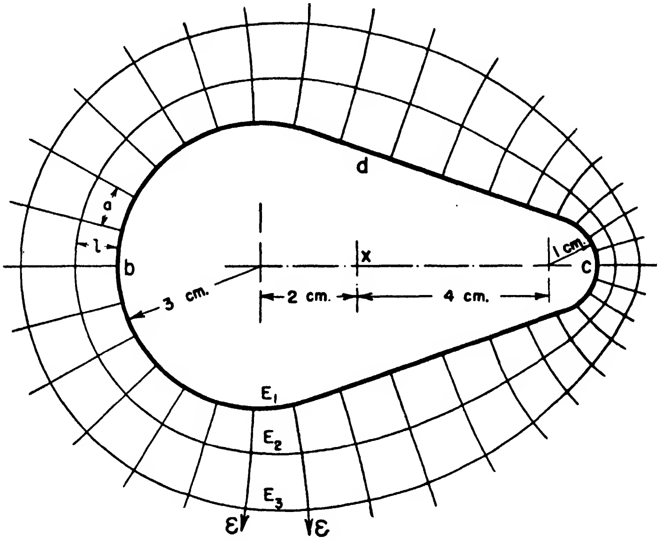


FIG. 25. Electric field map surrounding a long conductor of irregular cross-section. The cross-sectional view shown here illustrates how lower radii of curvature produce greater magnitudes of $\mathcal{E}$. See Prob. 9.

9. In Fig. 25 is shown a two-dimensional field map in the region adjacent to a conductor of somewhat irregular shape. If $E_1 = 1000$ volts, $E_2 = 600$ volts, and the distance l shown in the figure is 1.0 cm, find the magnitude of the $\mathcal{E}$ vector just outside the conductor at point b; also the magnitude of the $\mathcal{E}$ vector just outside the conductor at point c.

10. Assume that a long metal rod could be given an instantaneous uniform negative surface charge density, that is, an equal number of excess electrons per unit area over the entire surface. Explain, qualitatively, the nature of the redistribution of surface charge which would immediately set in. Under what condition would an electron originally at the center section come to rest before it reached the end of the rod?

11. What is the electric flux density in coulombs per square meter at $l = 2$ m in Fig. 24? Assume that the 5×10^{-7} coulomb charge is a small spherically distributed charge.

12. What is the electric flux density in coulombs per square meter and the corresponding value of $\mathcal{E}$ at a point 10 cm from the surface of a metal sphere which is charged with -3×10^{-5} coulomb? The diameter of the metal sphere is 10 cm, and $\epsilon_r = 1$.

13. A small metal object located not far above the earth's surface is deprived of one billion of its free electrons. Assuming that this metal object is the only body in the vicinity (except the earth), draw a two-dimensional sketch (not a map) showing the origination and the termination of the electric flux in the vicinity of the object. What amount of electric flux emanates from the metal object?

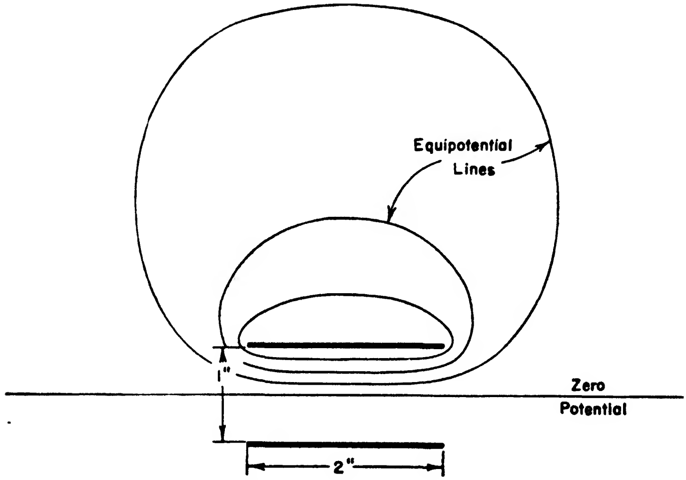


FIG. 26. Illustrating three equipotential lines which encircle a flat conductor. Completion of the map will show the relative amount of electric flux which threads its way from the upper surface of the upper conductor to the lower surface of the lower conductor. See Prob. 15.

14. Derive a general expression for the potential at a point midway between two small equal charges ($+Q$ each) which are spherically distributed charges separated center-to-center by a distance D. The midway point referred to is on a line joining the charge centers, and potential is to be reckoned relative to true or absolute zero.

15. In Fig. 26, the heavy lines represent the cross-sections of two long flat conductors which are maintained at a potential difference of 2000 volts and immersed in a large vat filled with transformer oil of the kind specified in Table 1, page 179.

The equipotential lines shown in the figure have been evaluated by methods which are somewhat more advanced than we care to consider here. Another and corresponding set of equipotential lines encircle the lower conductor but are not shown in Fig. 26. Complete the field map at least above the zero potential line and determine the approximate amount of electric flux which crosses an 8-sq in. section of the zero potential surface which is 1 in. in length normal to the plane of the page and 4 in. in length either side of the vertical center.

16. An electron is situated between two flat parallel plates and just outside the negative plate. The plates are separated from one another by 0.01 m of air and have a potential difference of 1000 volts.

(*a*) With respect to the positive plate as a zero datum plane, what potential energy does the electron possess?

(*b*) Starting from rest, the electron travels unimpeded to the + plate. What kinetic energy does the electron possess when it reaches the + plate; and what becomes of this energy when the electron is stopped by the plate?

(*c*) What is the duration of the travel time in part (*b*).

17. What is the electric flux density between two parallel plates which are separated from one another by 2 mm of glass ($\epsilon_r = 8$), if the plates have a potential difference of 500 volts? What is the total charge per plate if the plates are 40 cm x 80 cm in area? Assume uniform flux distribution; that is, neglect fringing.

18. What is the total energy stored in the electric field of Prob. 17?

19. What mutual force of attraction is developed on the parallel plates shown in Fig. 23 if the plates are 40 cm x 80 cm in area and energized with 1000 volts? ($\epsilon_r = 1$.)

20. The electrometer shown in Figs. 12 and 13 has a movable plate which is circular, and the r dimension is 7.5 cm. The plates are separated by a distance of $d = 0.2$ cm. What is the magnitude of the charge on the movable plate when 0.025 kg of *weight* is required to overcome the electrical attraction developed on the movable plate.

21. What voltage must be applied to the plates of the electrometer described in Prob. 20 to make it operate in the manner specified there.

22. What would be the maximum rated voltage allowable in connection with the parallel plates shown in Fig. 23 if air is the dielectric material and the specification for rated voltage is "the breakdown strength of the dielectric shall be twice rated voltage plus 1000 volts."

23. Assume that a solid copper sphere 1 cm in radius can be deprived of one free electron in every billion atoms of copper in which there are 8.4×10^{22} atoms per cubic centimeter. Find the potential gradient at the surface of the sphere and compare the value thus found with the breakdown strength of air. Calculate $\mathcal{E}$ on the basis of the copper sphere being the only body in the vicinity of a large and perfectly evacuated region. What would be the result of ejecting the charged sphere into air?

24. An $\mathcal{E}$ vector in air is known to impinge upon a surface of distilled water ($\epsilon_r = 80$) at an angle of 45°. Make a sketch showing the change in direction and magnitude of the $\mathcal{E}$ vector as it crosses the boundary from air to water, considering only changes caused by change in ϵ_r.

25. Refer to Fig. 27.

(*a*) What number of coulombs of electric flux crosses the unit length Gaussian (or enclosing) surface shown if the conductors are equally and oppositely charged with 1500 statcoulombs per foot of axial length? Assume that all data are converted to mks units and that unit length in Fig. 27 is 1 m. $R_1 = 1.0$ in.; $R_2 = 4.8$ in.; and $\epsilon_r = 1$.

Note: $\dfrac{\text{(No. of) coulombs}}{\text{statcoulomb}} = \dfrac{1}{3 \times 10}$

(*b*) What is the potential difference between the conductors calculated with the aid of equation (24)?

(*c*) What is the potential drop from $x = R_1$ to $x = R_2$ if only the $+\sigma$ line charge is considered present; if only the $-\sigma$ line charge is considered present? Apply the principle of superposition and compare with the result found in (*b*).

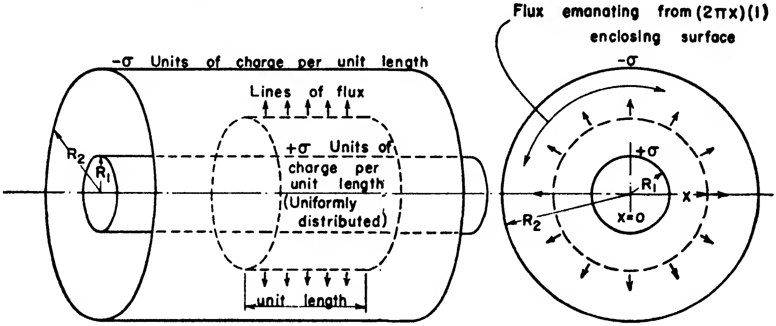


FIG. 27. Application of a Gaussian surface which encircles a unit length of the inner conductor of a coaxial cable. See Prob. 25.

26. Given two line charges as shown in cross-section in Fig. 20, page 185.

(*a*) Make a plot of the magnitude of the resultant $\mathcal{E}$ vector versus x from $x = r = 4$ mm to $x = D - r = 0.10$ m if the line wires are maintained at a potential difference of 1000 volts with air as the dielectric.

(*b*) Compare the $\mathcal{E} \times l$ area under the curve obtained in (*a*) with the known potential difference of 1000 volts.

27. What is the magnitude and direction (relative to the $+x$ direction) of the resultant $\mathcal{E}$ vector at a point which is 2 cm directly above the center of the $+\sigma$ line charge shown in Fig. 20, page 185, if the voltage difference between conductors is 1000 volts, $r = 4$ mm, $D = 10.4$ cm, and $\epsilon_r = 1$?

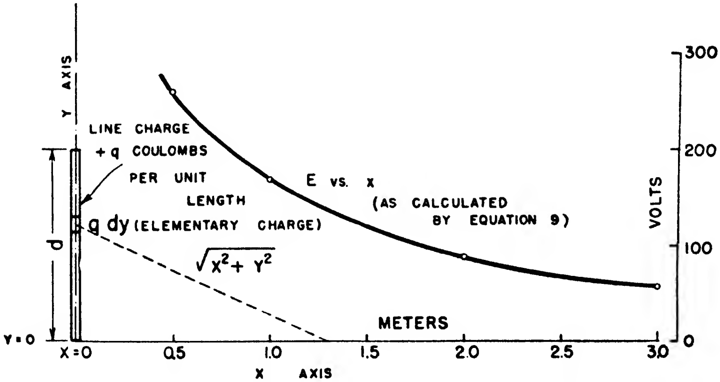


FIG. 28. Evaluation of potential near a distributed line charge. For points along the x axis the potential due to the elementary charge, $q\,dy$, $E_x = \dfrac{q\,dy}{4\pi\epsilon\sqrt{x^2 + y^2}}$. See Prob. 28.

28. In Fig. 28 is shown a uniformly distributed line charge which is $d = 1$ m in length. The charge which is presumably distributed over this length totals 2×10^{-8} coulomb, and the surrounding medium is air. Find the absolute potential at $x = 0.5$ m and at $x = 3.0$ m due to the distributed line charge which as shown runs from $y = 0$ to $y = d$: (*a*) either by deriving the general expression for the absolute potential at points along the x axis due to the line charge elements, or (*b*) approximating the correct result by lumping the entire charge at the point $y = d/2$, $x = 0$.

Note: The general expression for E_x referred to in (*a*) turns out to be

$$E_x = \frac{q}{4\pi\epsilon_0\epsilon_r} \ln \frac{(d + \sqrt{d^2 + x^2})}{x}$$

and to slide-rule accuracy the results obtained with this expression are $E_{0.5} = 260$ volts and $E_{3.0} = 58.8$ volts.

29. Two metallic spheres each 2 cm in diameter are equally and oppositely charged with 5×10^{-7} coulomb. These spheres are separated in air from one another by a center-to-center distance of 2 m.

What is the potential gradient at a point midway between the spheres on the line which joins the centers of the spheres?

30. What is the potential difference between the spheres described in Prob. 29?

31. Consider an electron which is initially at rest at the surface of the inner of two concentric cylindrical conductors that are energized with a potential difference of 300 volts, the inner conductor being negative. The ratio of the inner radius of the outer conductor to the radius of the inner conductor is 10. Inner conductor radius = 0.1 cm.

Find the length of time required for the electron to travel across the free-space medium between the conductors. Assume that the electron is unimpeded by any force but that of inertia.

CHAPTER VIII

The Magnetic Field

1. An Initial Concept. The physical facts illustrated in Figs. 1-a and 1-b will be considered briefly in order to introduce a concept which is fundamental to the study of the magnetic field. In considering Fig. 1-a, for example, the two current-carrying conductors which are shown deformed under the mutual forces of attraction might be thought of as being less rigid than the other parts of the two circuits since all current-carrying elements in both circuits experience forces of a similar nature.

The two current-carrying loops of Fig. 1-a are known experimentally

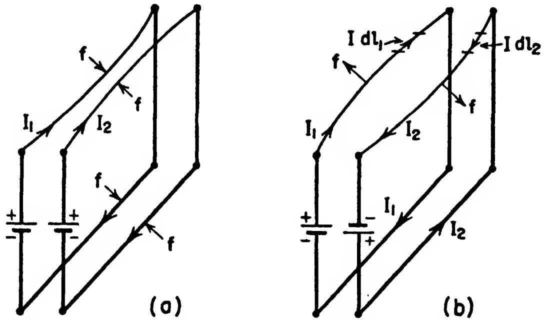


FIG. 1. Current-carrying conductors experience mutual forces of attraction or repulsion.

to experience mutual forces of attraction, and the two loops shown in Fig. 1-b are known experimentally to experience mutual forces of repulsion. Bus bars and heavy conductors in large machines must be well braced to prevent deformation when short-circuit currents flow as they do when faults like line-to-line contacts and undesired grounds appear on the system.

To begin with, it should be realized that the forces shown in Fig. 1 are essentially due to the *movement of charge I*. The magnitudes of the forces developed by the electric field (that is, by the $\mathcal{E}$ vector) in a case of this kind are insignificantly small compared to the forces which are produced by the moving charges, namely, by I. We might, for example,

open the circuits shown in Fig. 1-a and let the batteries charge the wires electrostatically over the entire lengths of the wires, and the electric forces developed in this low-voltage system would not be comparable in magnitude to the forces which are developed when current is permitted to flow through the conductors.

We are, therefore, confronted with forces which act on *moving* charges and which cannot be accounted for in terms of electric field theory. It is only in terms of magnetic field theory that the forces illustrated in Fig. 1 and various other electromagnetic phenomena can be explained. In the early development of magnetic field theory, we shall employ current-carrying loops (or turns) like those shown in Fig. 1 as the origin or source of the magnetic field, in much the same manner as we employed stationary charges as the origin of the electric field.

Since the magnetic field established by a current-carrying loop depends upon the magnitude of the current I and upon the length of the loop l, we shall symbolize the source of the magnetic field as Il or as $\oint I\,dl$. The loop around the integral sign implies only that the $I\,dl$ elements of which the current-carrying loop is composed are to be considered as acting around the entire loop. In symbolizing the source of a magnetic field as $\oint I\,dl$, we are not implying that any actual integration need be performed. We are implying, however, that the source of magnetism is dimensionally the *product of current and distance*, two physical quantities that are well established.

It will be shown presently that a current-carrying loop has all the properties that are to be expected of a magnetic field source, and later it will be shown that a magnetic field encircles or links with *any* current path.

2. Magnetic Poles as Sources of Magnetic Lines. The commonly accepted version of a magnetic field is that a magnetic field is any region in which an ordinary compass needle is acted upon by forces that tend to align the S-to-N axis of the needle in a particular direction as illustrated in Fig. 2. A continuous path which is defined in direction by the S-to-N axial direction of a small exploring compass of the type shown in Fig. 2 is called a *magnetic line*. This is the only physical significance which need be attached to a *magnetic line* for the present. After it has been shown that current loops are the equivalents of magnetic poles, the magnetic line as used here will be replaced with a well-defined magnetic vector; but in the early stages of the development of magnetic field theory, the resultant force which acts on a compass needle to align it in the magnetic field is our most tangible asset.

The small region near the end of a bar magnet, from which most of the *magnetic lines* appear to emanate, is called a north pole, and the small region near the other end, into which these lines appear to enter the magnet, is called a south pole.

Even though a magnetic pole is a poorly defined region near the end of a magnet and even though a north pole must always be complemented

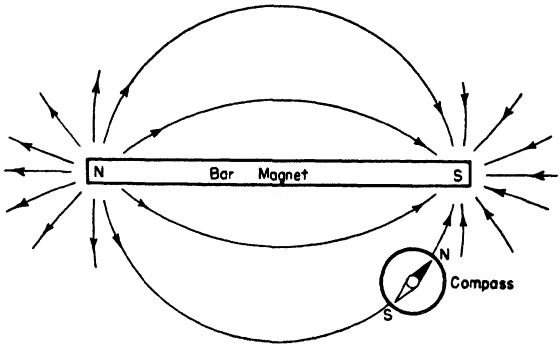


FIG. 2. Exploring the magnetic field established by a bar magnet with the aid of small compass needle.

with a south pole, Coulomb, employing long slender bar magnets and a torsion balance, was able to deduce the following:

$$f = \mu_0 \frac{M_1 M_2}{4\pi r^2} \quad \text{(in rationalized units)} \tag{1}$$

where f is the mutual force of attraction or repulsion between M_1 and M_2
M_1 and M_2 are the pole strengths of the apparent sources of magnetic lines
μ_0 is the permeability of free space, the numerical value of which is $4\pi \times 10^{-7}$ in rationalized mks units
r is the center-to-center separation of M_1 and M_2.

We realize, of course, from our experience with bar magnets that like sources (or like M's) repel one another and that unlike sources (or unlike M's) attract one another.

The experimentally determined relationship given in equation (1) is useful to the extent that it tells us what to expect from a source of magnetism. Pole strength M will not be classed as a new or undefined quantity since all experimental evidence points to the fact that current flowing in a closed loop $\left(\oint I\, dl = Il\right)$ is actually the source of mag-

netism. Dimensionally then

$$M = I^1 l^1$$

and from equation (1)

$$\mu_0 = f^1 l^2 M^{-2} = f^1 l^2 (Il)^{-2} = f^1 I^{-2} \tag{1-a}$$

μ_0 is the property of free space (or air) which permits magnetic forces of attraction and repulsion to manifest themselves through free space. Although the dimensional expression for μ_0 given in equation (1-a) is rather meaningless from a physical point of view, it will be useful later when magnetically developed forces and magnetically generated voltages are investigated.

The Product $\mu_0\epsilon_0$. As an artifice for remembering the numerical value of either μ_0 or ϵ_0 if one of them is known, we have the following relationship which holds in any systematic set of units:

$$\mu_0\epsilon_0 = \frac{1}{c^2}$$

where c is the velocity of light expressed in the system of units being used. Without attempting to show just why the product $\mu_0\epsilon_0$ should turn out to be the reciprocal of the velocity of *light* squared, we shall show from Coulomb's inverse square laws that we should expect the product $\mu_0\epsilon_0$ to be the reciprocal of *some* velocity squared. From these two laws,

$$f = Q^2 l^{-2} \epsilon_0^{-1} \quad \text{for electric forces [from equation (1), Chapter II]}$$

$$f = M^2 l^{-2} \mu_0^1 = I^2 \mu_0^1 \quad \text{for magnetic forces [from equation (1-a)]}$$

Equating the right-hand sides of these dimensional expressions, we have

$$\mu_0^1 \epsilon_0^1 = Q^2 l^{-2} I^{-2} = Q^2 l^{-2} Q^{-2} t^2 = l^{-2} t^2 = v^{-2}$$

Hence the product $\mu_0\epsilon_0$ is shown to be dimensionally the reciprocal of velocity, $1/v$, squared. In rationalized mks units,

$$\mu_0\epsilon_0 = (4\pi \times 10^{-7}) \left(\frac{1}{36\pi \times 10^9}\right) = \frac{1}{9 \times 10^{16}} = \frac{1}{c^2}$$

In unrationalized mks units,

$$\mu_0'\epsilon_0' = (10^{-7}) \left(\frac{1}{9 \times 10^9}\right) = \frac{1}{9 \times 10^{16}} = \frac{1}{c^2}$$

3. Current-Carrying Loops as Sources of Magnetic Lines. Let us observe the result of replacing the north and south poles of the bar magnet and compass shown in Fig. 2 with current-carrying loops as indicated in Fig. 3. To begin with, we must realize that a current-carrying loop is *two-faced*; viewing from in front of one face an observer sees current directed in a counterclockwise direction; for example, if he

looks at the bar magnet from the observer position shown in Fig. 3. But this same loop, if viewed from the opposite face, would appear to have current flowing around the bounded surface, clockwise in direction. Two definitions which take into account the two-faced nature of a current-carrying loop are:

(1) A north pole is that side of a surface (which is bounded by current) around which an observer sees counterclockwise directed current.

(2) A south pole is that side of a surface (which is bounded by current) around which an observer sees clockwise directed current.

A current-carrying loop may be *either* a north pole or a south pole depending solely upon how it is viewed by an observer who is directly in front of one face of the loop. An outside or more aloof observer, of

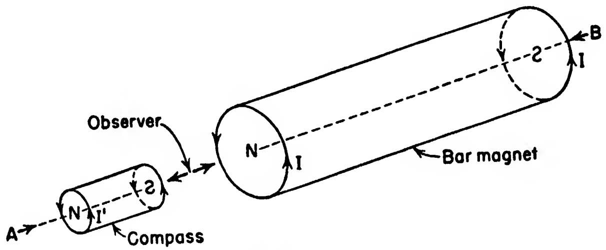


FIG. 3. Current-carrying loops as sources of magnetism.

course, sees the current-carrying loop *both* as a north pole and as a south pole. It is fundamental to all magnetic circuit theory that *magnetic lines close on themselves*, that is, any one line is continuous. A magnetic line which appears to emanate from one face of a current-carrying loop returns to the other face and in so doing completes a circuital path which renders it continuous. This is somewhat different from an electric field line which is defined in direction by the direction which a test charge would follow in an electric field. Whereas an electric field line (or flow line) originates at a positive charge and terminates at a negative charge, a magnetic line (or flow line) is continuous and must always *link* with the closed path of the moving charge which produces it, the linking being in the sense that neighboring elements of a chain link with one another.

A careful study of Fig. 3 will show that an observer at the indicated position sees a clockwise-directed current (or a south pole) when he looks toward S of the compass, and that he sees counterclockwise-directed current (or a north pole) when he looks at N of the bar magnet. Thus two current loops carrying oppositely directed currents (to an

observer between the loops) experience mutual forces of attraction. (This conclusion agrees with the known physical facts shown in Fig. 1-a.) Two current loops carrying similarly directed currents (to an observer between the loops) experience mutual forces of repulsion in agreement with the physical facts illustrated in Fig. 1-b.

The permanent magnets shown in Fig. 2, the bar and the compass, may be replaced as shown in Fig. 3 with equivalent current-carrying loops which for the present are merely closed paths around which electric charge is in motion. Experiments of an advanced nature[1] actually indicate that permanent magnetism is produced by electrons spinning on their own axes in somewhat the same manner as the earth spins on its axis of rotation. The picture is not quite this simple but in an elementary way we can readily visualize that any space-distributed charge which is spinning about some axis of rotation constitutes a current loop (or several current loops in parallel) since charge in motion around a closed path is our only requirement for the current loop which we have symbolized as $\oint I\,dl$. With this concept in mind, we shall later find that ferromagnetism and permanent magnetism are not basically different in origin from the magnetism which is produced by current-carrying coils.

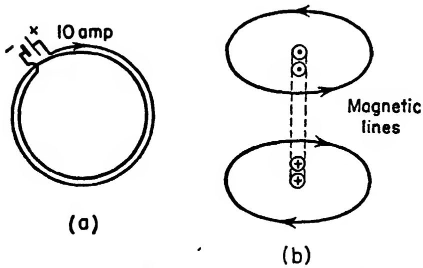


FIG. 4. A magnetomotive force establishing magnetic lines which link with the current paths.

Even though $\oint I\,dl$ is a measure of the magnetism which is produced by a current-carrying coil, $\oint I\,dl$ is not employed directly in the evaluation of this magnetism for reasons which will become apparent later.

4. Magnetomotive Force $\mathcal{F}$. The source of magnetism is called magnetomotive force, abbreviated mmf, and symbolized by $\mathcal{F}$. Magnetomotive force is customarily measured in ampere-turns (NI) without regard to the length of the turns. For the two-turn coil shown in Fig. 4, for example, the magnetomotive force is

$$\mathcal{F} = NI = 2 \times 10 = 20 \text{ amp-turns} \quad \text{(of mmf)} \tag{2}$$

The *ampere-turn* is the primary unit of mmf in rationalized mks units, and, although dimensionally the same as the ampere, the ampere-turn

[1] "The Physical Basis of Ferromagnetism," by R. M. Bozorth, *The Bell System Technical Journal*, Vol. XIX, No. 1.

signifies a magnetic property of the current-carrying coil which is usually not associated with the ampere itself.

Magnetomotive force is to the magnetic circuit (or field) what electromotive force is to the electric circuit (or field). As such magnetomotive force is called magnetic potential *rise*, and it acts across the faces of a current-carrying coil directed *from* the south pole face *to* the north pole face. The positive direction of the magnetic potential rise through the faces of a coil defines the positive direction of the magnetic lines. (See Fig. 4-b.)

The Right-Hand Rule. The determination of the positive direction of the magnetic lines (or of the magnetic potential rise through the faces of a current-carrying coil) may be readily made in terms of the *right-hand rule.* If the fingers of the right hand are thought of as encircling the current-carrying coil in the direction of $+I$, the thumb points in the positive direction of the magnetic lines.

As applied to Fig. 4-a, the $+I$ direction is clockwise (indicating a south pole), and an application of the right-hand rule specifies the magnetic lines as being directed into the plane of the page.

In Fig. 4-b, where a cross-sectional view of the two-turn coil is illustrated, the $+$ and $\cdot$ symbols indicate the directions of the current, either into the page in the case of $+$ or out of the page in the case of $\cdot$. (This symbolization may be easily remembered if one thinks of $+$ as the feathered tail of an arrow going away from the observer and the $\cdot$ as the point of an on-coming arrow.) With the aid of these symbols, the direction of $+I$ around the coil is readily envisioned, and the application of the right-hand rule becomes a routine procedure.

Cases will arise where a portion of a current loop in the form of a long straight conductor will be considered the source of magnetic lines. In such a case the magnetic lines encircle the conductor having a positive direction which agrees with the direction of the fingers of the right hand if the thumb is pointed along the wire in the direction of $+I$.

5. Magnetic Potential Drop, U. We have considered mmf ($\mathcal{F}$) as a well-defined and easily calculated source of magnetic lines, and we know from the fundamental circuital nature of these magnetic lines that this mmf projects its effects around the various closed paths which link with NI in the sense previously referred to. It is for this reason that magnetomotive force is considered as magnetic potential *rise*, analogous to electromotive force (or electric potential *rise*) in an electric circuit. That is, NI as we have evaluated it in the preceding section may be considered the *battery* of the magnetic paths or circuits which link with NI.

We may further consider that *any* closed path which links with NI amp-turns of mmf (or magnetic potential rise) consumes this mmf in the

same way that a dissipative electric circuit consumes the emf (or electric potential rise) of the electric circuit. Then around any closed path which links with NI mmf there must be a corresponding amount of magnetic potential drop which we shall here symbolize as U to designate the fact that we are speaking of the *magnetic potential drops* which appear along the closed paths that link with NI.

Kirchhoff's emf law applied to the magnetic source and to the magnetic circuit along which the magnetic potential drops occur would then read

$$\mathcal{F} - U = 0 \quad \text{or} \quad \mathcal{F} = U \tag{3}$$

After means have been found for evaluating U from known physical data (like voltage drops $= RI$ in the electric circuit), equation (3) will become a very important relationship because practically all the strictly magnetic problems encountered in engineering practice are solved with the aid of this equation.

6. The H Vector Defined as $\left.\dfrac{dU}{dl}\right]_{\text{max.}}$ · Let attention be focused on the arrow direction of one of the magnetic lines shown in either Fig. 4 or Fig. 5. As we proceed in the arrow direction we encounter *drops* of magnetic potential in much the same manner as we encounter voltage drops in the dissipative portions of an electric circuit as we proceed in the $+\mathcal{E}$ direction around an electric circuit. Where an $\mathcal{E}$ vector is defined in electric field theory as $\left.\dfrac{dV}{d\mathbf{l}}\right]_{\text{max.}}$, the **H** vector in magnetic field theory is defined as

$$\mathbf{H} = \left.\frac{dU}{d\mathbf{l}}\right]_{\text{max.}} \quad \left(\text{or simply } H = \frac{\Delta U}{\Delta l}\right) \tag{4}$$

where the subscript max. implies that $\mathbf{l}$ is to be directed *along* the magnetic lines which would be mapped out with an exploring compass. (See Fig. 2 or Fig. 5.) The primary unit of H in rationalized units is plainly ampere-turns per meter.

Where we know ΔU, say between two magnetic equipotential planes which are separated by the distance Δl, we calculate the average value of H over this distance simply as $\Delta U/\Delta l$; but before doing so we shall consider briefly what is meant by a magnetic equipotential plane.

By definition, a surface, all points of which are pierced *normally* by **H** vectors, is a magnetic equipotential surface. Although these surfaces are of importance in graphical or analytical mapping of magnetic fields, they are little used in magnetic *circuit* theory or practice. They lack the uniqueness possessed by electric equipotential surfaces as will be demonstrated.

Let it be required to find the magnetic equipotential planes in the magnetic field surrounding the long straight conductor shown in Fig. 5. The complete current-carrying loop is not shown in Fig. 5 because the return conductors are presumably far removed from the straight portion of conductor which is under investigation. We shall assume, however, that the straight conductor is ultimately bent to the right at both top and bottom ends of the long straight portion in such a manner that the north pole face of the current-carrying loop is about in the position shown by the N markings on the exploring compasses shown in Fig. 5. It will be remembered that magnetic lines emanate out of the north pole face of a current-carrying loop.

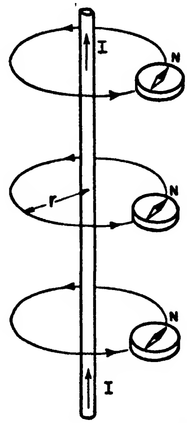


Fig. 5. Exploring the magnetic field near a long straight conductor.

From the mapping process indicated in Fig. 5, it is clear that the surfaces which will be pierced at right angles by the **H** vectors (or compass lines) will be planes which pass through the axis of the straight conductor as, for example, planes U_A and U_B shown in Fig. 6. These planes will, of course, extend vertically as far in either direction as the straight conductor can be considered to be *far* removed from the other current-carrying elements of the current loop.

From equation (4), we deduce that $H\,\Delta l = \Delta U$. This means that if we follow a known **H** vector for a distance Δl we shall encounter a specified number of ampere-turns of magnetic potential drop. The logical starting place to begin counting $H\,\Delta l$ drops would appear to be the north pole face of the current-carrying coil, say the U_A plane shown in Fig. 6. As we proceed to the U_B plane (by way of *any* path) we shall encounter $H_1l_1 (= H_2l_2)$ ampere-turns of magnetic potential drop; and if we proceed in a circular path of r radius completely around the conductor as indicated in Fig. 5 we shall encounter $2\pi rH_r$ amp-turns of magnetic potential drop. (The details of this calculation, which are very simple, will be given in the following section.)

Now the plane or surface which is presumably $2\pi rH_r$ ampere-turns *below* the starting plane in magnetic potential actually occupies the same position in space as does the surface from which we started.[2] It is

[2] In this respect, magnetic equipotential surfaces are different from electric equipotential surfaces. Two electric equipotential surfaces (of different potentials) can never occupy the same position in space nor can they cross one another. Both sides of an electric equipotential surface are at the same potential, whereas the two sides of a surface in a magnetic field *may* differ in potential by the amount of the mmf which acts through this surface.

possible, however, to avoid ambiguity in most cases if we consider that a surface in a magnetic field which lies in the plane of a two-faced current-carrying loop is composed of two equipotential faces, one of which is higher in magnetic potential than the other by the mmf rise produced

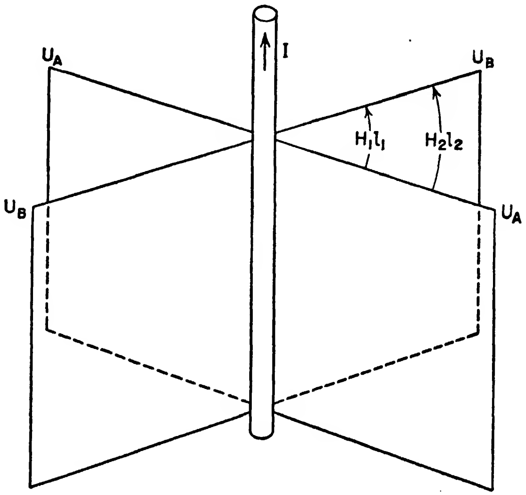


FIG. 6. U_A and U_B are magnetic equipotential planes of the magnetic field mapped in Fig. 5.

by the current loop. As applied to Fig. 6, this means that we consider the north pole face of U_A as being I amp-turns higher in potential than the south pole face of U_A, since a single-turn loop is involved. Under

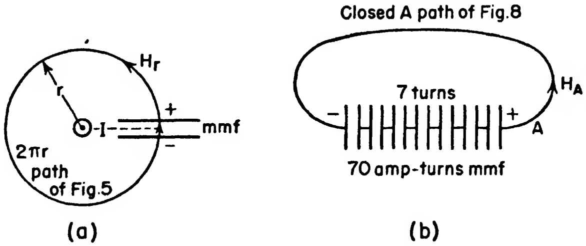


FIG. 7. Magnetic circuits of a single wire and of a 7-turn coil.

these conditions it is possible to represent the magnetic circuit as shown in Fig. 7-a which is recognized at once as being analogous to the electric circuit if we consider NI (I in this case) to correspond to the battery or generator of the electrical circuit. A north pole face of an mmf corresponds to the + terminal of the battery and the south pole face to the − terminal.

Without the aid of the concepts presented in Fig. 7 we are sometimes in doubt just how or where the mmf gets *into* the magnetic circuit; as, of course, it must if we are to employ relationships like that stated in equation (3) in the solution of magnetic circuit and field problems. The concepts presented in Fig. 7 are simple and straightforward provided the (mmf = I) which is developed by a current-carrying loop (between its south pole face and its north pole face) is entered into the magnetic circuit wherever the circuit or path crosses the faces of a current-carrying loop. The basic concept may, of course, be extended to include as many current loops as are actually present in any particular case, as, for example, the seven loops of Fig. 8 which are reduced to circuital form in Fig. 7-b. Across each current-loop face, I amp-turns of magnetic potential rise are developed.

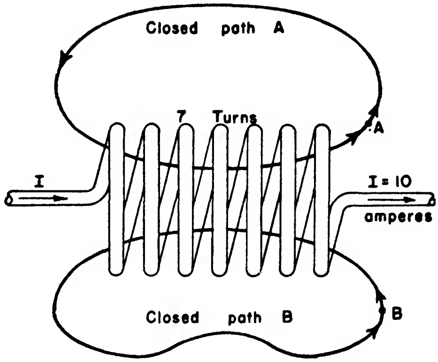


FIG. 8. The magnetic potential drop once around either path A or path B is 70 amp-turns.

7. The Circuital Law of Magnetism. Equation (3) may now be written in a more significant manner:

$$\underset{\text{mmf}}{\mathcal{F}} = \underset{\text{mag.pot. drops}}{\sum H\,\Delta l} \quad \text{(around any closed loop)} \tag{5}$$

or more elegantly as

$$\mathcal{F} = \oint H \cos \theta]_{\mathrm{H}}^{dl}\, dl \quad \text{[from equation (4)]} \tag{6}$$

where the small circle around the integral sign indicates that the integration (or the counting up of $H\,dl$'s) is to be carried out around a closed path. If this path links with an mmf, $\mathcal{F}$ has a finite value equal to the NI linked. In case the path selected links with no current loops (in the sense that neighboring elements of a chain link), $\mathcal{F}$ in equations (5) or (6) is zero since we have no *magnetic battery* in the circuit.

Either equation (5) or equation (6) represents *the circuital law of magnetism.* This law is basic to all magnetic circuit problems because this mmf law of magnetism occupies the same position in magnetic theory that Kirchhoff's emf law occupies in electric theory. The elegant form shown in equation (6) should not be construed to mean that the

operations involved are necessarily difficult to perform. Several examples of a basic nature will be given to show the ease with which **H** can be calculated in cases involving cylindrical symmetry.

Even though the path taken in a magnetic field does not follow a magnetic line as, for example, the closed path B in Fig. 8, this path is nevertheless acted upon or energized with NI ampere-turns of mmf if it links with this amount of mmf. Where the path selected does not follow a magnetic line, $\cos \theta]_{\mathbf{H}}^{dl}$ in equation (6) has some value other than unity because in this case dl considered as a vector is not directed along the same path as the vector **H**. For purposes of calculations, we almost always select a path in a magnetic field which coincides with the **H** direction because in so doing we not only make $\cos \theta]_{\mathbf{H}}^{dl}$ equal to unity in equation (6) but we have a path which can usually be followed with more ease than can an irregular path; assuming, of course, that we know from symmetry or otherwise the direction of the **H** vectors in the magnetic field. If the magnetic (or **H**) lines cannot be visualized, field mapping or detailed point-by-point calculations of a kind to be considered later must be resorted to in order to find the directions of the magnetic lines.

Example 1. *Single Straight Conductor* **H**'s. Let it be required to evaluate **H** at a radial distance r from the straight conductor shown in Fig. 5, the top view of which is illustrated in Fig. 7-a in circuital form.

Owing to the cylindrical symmetry which exists, we know that **H** is constant in magnitude at any point along the $2\pi r$ path. The application of equation (6) to this simple case will show that

$$\mathcal{F} = H_r \oint dl$$

where $\mathcal{F} = I$ (thė current linked by the $2\pi r$ path)

$\oint dl = 2\pi r$ (the circumference of the circular path).

Hence

$$I = 2\pi r H_r \quad \text{or} \quad H_r = \frac{I}{2\pi r} \quad \text{amp-turns/m} \tag{7}$$

where the direction of the **H** vector is plainly along the circular path.

Equation (7) is sometimes referred to as the Biot-Savart law, after the men who discovered this fact experimentally and proved that **H** is constant in magnitude around any circular path of r radius about a long straight current-carrying conductor.

It will be observed from equation (7) that the magnitude of the **H** vector decreases as the radial distance r is increased. The planes of equal magnetic potential in this field are shown in Fig. 6, and they have been discussed in the preceding section. A two-dimensional magnetic field map of the

region in the vicinity of the conductor can readily be constructed with the aid of a compass and a straightedge.

Example 2. *Parallel-Wire Transmission Line* **H**'*s.* Let it be required to evaluate the resultant **H** at any point between two long current-carrying conductors (such as shown in cross-section in Fig. 9) on a line joining the centers of these conductors.

The result obtained in Example 1 may be applied to each of these long conductors in turn; then the resultant magnitude (of **H**) may be obtained simply as $\mathbf{H}_x + \mathbf{H}_y$ since these component vectors are similarly directed along the straight line joining the conductor centers. Thus

$$H_x = \frac{I}{2\pi x} \qquad H_y = \frac{I}{2\pi y} \qquad H_{\text{res.}} = \frac{I}{2\pi}\left(\frac{x+y}{xy}\right) \quad \text{amp-turns/m} \tag{8}$$

where the x origin is at the center of the I_{out} conductor of this transmission line and the y origin is at the center of the I_{in} conductor, as shown in Fig. 9.

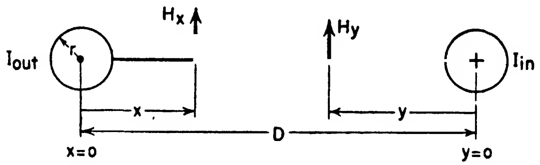


FIG. 9. For calculating **H**'s between conductors of a parallel-wire line.

The plotting of the resultant value of H between the two wires is left as a student exercise at the close of the chapter.

Example 3. *Interpretation of a Magnetic Field Map.* The complete map of the magnetic field in the immediate vicinity of two current-carrying wires is the same as that given in Fig. 21, page 187. In viewing this map as a magnetic field map we see the **H** paths as the circular paths which enclose the conductors. We visualize the straight line joining the conductors as the north pole face *and* as the south pole face of the current loop which is linked by the **H** paths. The mmf or magnetic potential *rise* from, say, the *bottom* of this line to the *top* of this line is in this case I amp-turns, and this amount of mmf projects itself into space in the form of $H\,\Delta l$ magnetic potential drops along each of the **H** paths of which there are eleven shown in Fig. 21, page 187. The straight **H** path shown meets at infinity to meet the requirement that all **H** paths close on themselves. The planes of equal magnetic potential are shown in cross-section in Fig. 21 by the curved lines joining the two conductors, and it should be realized the same value of $H\,\Delta l$ magnetic potential drop exists between any two adjacent lines regardless of their physical separation. The greater the value of Δl between these lines, the less the magnitude of **H** in order to maintain the equality of $H\,\Delta l$ between the lines.

The important lesson to be gained from this example is that *all* the **H** paths which link with the I amp turns of mmf established by the current loop (that forms the transmission line) experience precisely the *same mmf* as they cross

the straight line joining the conductors. Hence all the **H** paths shown in Fig. 21, page 187, may be considered to be in parallel and energized with the same *magnetic battery*.

8. Coaxial Cable H's. In the cross-section of the coaxial cable transmission line shown in Fig. 10, it will be assumed that I amp of current flow in each conductor, the current in the inner conductor being directed out of the page to produce **H** paths as indicated and the current in the outer conductor being directed into the page. It will be observed that one conductor of the current loop completely encircles the other conductor in this type of transmission line. As a result of this geometrical configuration, there is no particular surface which can be designated as the seat of mmf.

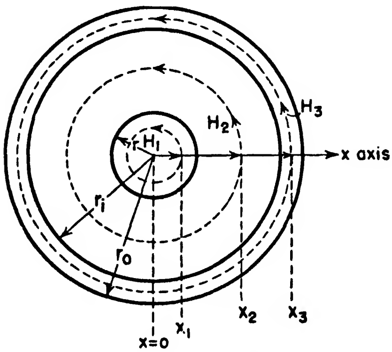


FIG. 10. For calculating H's in a coaxial cable.

The mmf which produces any of the **H**'s shown in Fig. 10 is equal to the current which is encircled by the **H** path, and this mmf may be thought of as distributed throughout the entire length of the **H** path. In other words, a finite distance exists in the +**H** direction between the south pole face and the north pole face of the *magnetic battery*, the distance being the circumference of the circular **H** path. It is as if this distributed battery were short-circuited.

Several different values of H are present in the coaxial cable, and they will be calculated in the order in which they appear, the center of the inner conductor being the starting point. In each case the circuital law of magnetism will be employed in the evaluation of H.

(a) Inside the inner conductor where $x < r$, the $\mathcal{F}$ in equation (6) is only $(\pi x_1^2/\pi r^2)\, I$; uniform current density is assumed over the cross-sectional area of the conductor. Therefore

$$\frac{x_1^2}{r^2} I = 2\pi x_1 H_1 \quad \text{or} \quad H_1 = \frac{x_1 I}{2\pi r^2} \quad \text{amp-turns/m} \tag{9}$$

provided $x_1 < r$. It will be observed that *inside* the inner conductor of the cable **H** increases linearly in magnitude as x_1 (the distance from the center) increases.

(b) In the region between the two conductors shown in Fig. 10, the value of H_2 is plainly

$$H_2 = \frac{I}{2\pi x_2} \quad \text{amp-turns/m} \tag{10}$$

since the $2\pi x_2$ circular path in this region links with all the current of the inner conductor but none of the current of the outer conductor.

(*c*) In the region occupied by the outer conductor (where $r_i < x < r_o$) in Fig. 10, the value of H_3 is obtained from equation (6) by noting that a $2\pi x_3$ circular path links with or has *within* its circumference an mmf of

$$\underset{\text{inner } i}{I} - \underset{\text{outer } i}{\frac{\pi x_3^2 - \pi r_i^2}{\pi r_o^2 - \pi r_i^2} I} = \frac{(r_o^2 - x_3^2)}{(r_o^2 - r_i^2)} I \quad \text{amp-turns}$$

since the current in the outer conductor is oppositely directed to the current in the inner conductor and only partially linked by the $2\pi x_3$ path. Substituting values into equation (6) as before,

$$H_3 = \frac{(r_o^2 - x_3^2)\, I}{2\pi x_3\, (r_o^2 - r_i^2)} \quad \text{amp-turns/m} \tag{11}$$

provided x_3 is restricted to values not less than r_i, the inner radius of the outer conductor, in keeping with the derivation given.

It will be observed that at $x_3 = r_o$, $H_3 = 0$. This is in keeping with the fact that outside the outer conductor of the coaxial cable there is present no magnetic field which is due to the conductor currents. If equation (6) were applied to a circular path where $x > r_o$, the mmf linked by this path would be zero since both $+I$ and $-I$ would be enclosed within the path.

9. Evaluation of H Employing Ampere's Law. Except in special cases like those involving cylindrical symmetry, **H** is not constant in magnitude throughout the entire length of the magnetic line along which **H** is directed. The parallel-wire line considered in Fig. 9 is an example, but in this case the problem of finding **H** was solved (in a restricted sense) by applying the principle of superposition to the cylindrical fields established by each of the long conductors.

The evaluation of the vector **H** at *any* point in the magnetic field (say at point p) is accomplished (at least formally) by:

(1) subdividing all the current-carrying conductors in the neighborhood of point p into small ($I\, d\mathbf{x}$) elements as indicated in Fig. 11;

(2) finding the incremental or, better yet, the infinitesimal value of **H** due to each ($I\, d\mathbf{x}$) element, namely, $d\mathbf{H}_p$ (at point p);

(3) summing up all the $d\mathbf{H}_p$'s at point p (in vector form) to find $\mathbf{H}_p$.

In a general case the procedure outlined above represents a staggering amount of work and is usually not attempted except along certain lines or planes which are so symmetrically located to all the ($I\, d\mathbf{x}$) elements

in the neighborhood that arithmetical summation of the $d\mathbf{H}_p$'s is possible.

The method to be employed is based upon an experimentally determined relationship known as Ampere's law which reads

$$dH_p = \frac{(I\,dx)\sin\theta\big]_l^{(I\,d\mathbf{x})}}{4\pi l^2} \quad \text{amp-turns/m} \tag{12}$$

where dH_p is the infinitesimal free-space value of H at point p

$(I\,dx)$ is an infinitesimal length of current-carrying conductor

l is the straight-line distance between $(I\,dx)$ and point p

$\sin\theta\big]_l^{(I\,d\mathbf{x})}$ is the sine of the angle between $\mathbf{l}$ and $(I\,d\mathbf{x})$ extended.

A careful study of equation (12) and Fig. 11 will show that only the component of $(I\,d\mathbf{x})$ which is perpendicular to $\mathbf{l}$ is effective in establishing

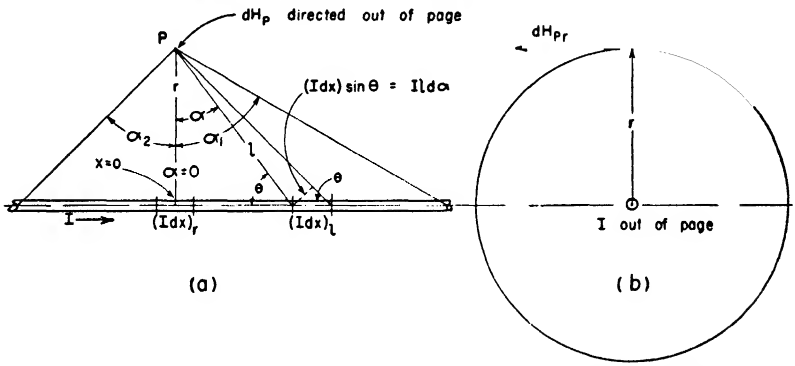


FIG. 11. Each $(I\,d\mathbf{x})$ element contributes to the resultant $\mathbf{H}$ at point p.

dH_p; and that the direction of dH_p (considered as a vector) due to *any* $(I\,d\mathbf{x})$ element is circularly directed *around the axis* of $(I\,d\mathbf{x})$ considered as a vector. It is expected that some thought may be required on the part of the reader before the full implications of these statements are evident to him. We shall, however, proceed on the basis that Ampere's law as stated is correct because it has been checked in the light of critically conducted experiments and because in any known case in free space (or air) equation (12) yields answers which agree with the circuital law of magnetism. Obviously, no one ever measured dH_p which is due to a single $(I\,dx)$ current-carrying element.

Referring to Fig. 11, it will be seen that, in the plane which is at right angles to the direction of $(I\,d\mathbf{x})$,

$$dH_{pr} = \frac{(I\,dx)_r}{4\pi r^2} \quad (\text{since } \sin\theta = \sin 90°)$$

and dH_{pr} is directed tangent to the circle (of radius r) which lies in the plane. See Fig. 11-b. The right-hand rule previously discussed may be employed to find the positive direction of $d\mathbf{H}_p$ around the circle.

For points which are not in the plane normal to the direction of $(I\, d\mathbf{x})$, the expression given for dH_p in equation (12) is rather awkward to handle, so we change from the variable x to the variable α as shown in Fig. 12 and in so doing replace $(dx \sin \theta)$ in equation (12) with its equivalent value $(l\, d\alpha)$. This change of variable will in some cases make the work of summing up the dH_p's much easier because Ampere's law then reads

$$dH_p = \frac{(Il\, d\alpha)}{4\pi l^2} = \frac{(I\, d\alpha)}{4\pi l} \tag{13}$$

In this form, the summing up of all the dH_p's due to all the $(I\, dx)$ elements along the straight wire shown in Fig. 11 is an easy matter.

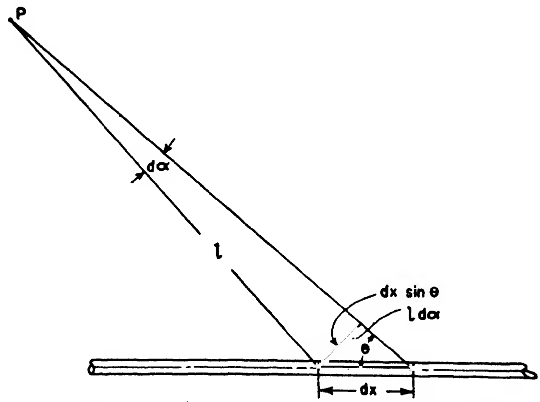


FIG. 12. Showing that $dx \sin \theta = l\, d\alpha$ in the limit as $dx \to 0$.

Example 1. *The Short Straight Conductor.* Let it be required to find the magnitude of $\mathbf{H}_p$ at point p in Fig. 11 due to a relatively short section of conductor as shown in the illustration. The total length of conductor will be taken such that the left end of the conductor subtends an angle $-\alpha_2$, and the right end of the conductor subtends an angle $+\alpha_1$ as measured from point p; the $\alpha = 0$ origin is as shown in Fig. 11.

Using equation (13) and recognizing that I is not a function of α and that $r/l = \cos \alpha$, we have

$$H_p = \frac{I}{4\pi} \int_{-\alpha_2}^{\alpha_1} \frac{d\alpha}{l} = \frac{I}{4\pi r} \int_{-\alpha_2}^{\alpha_1} \cos \alpha\, d\alpha$$

or

$$H_p = \frac{I}{4\pi r} \Big[\sin \alpha \Big]_{-\alpha_2}^{+\alpha_1} = \frac{I}{4\pi r} (\sin \alpha_1 + \sin \alpha_2) \quad \text{amp-turns/m} \tag{14}$$

Equation (14) might, for example, be applied successively to the four sides of a rectangular shaped loop to find the magnitude of **H** at the center of the loop, where for any one side $\alpha_1 = \alpha_2$ and r is the perpendicular distance from the center of this side to the center of the rectangular loop. This problem will be left as an end-of-chapter exercise.

If the section of conductor under consideration is very long compared with r, and if point p is taken not too close to one end, both α_1 and α_2 in equation (14) approach $\pi/2$ and equation (14) reduces to the long-wire case given in equation (7).

Example 2. *The Circular Conductor.* Refer to Fig. 13. Let it be required to find the magnitude of **H** at any point along the axis of the circular loop. It will be observed that only one component of the dH_p produced by any $(Ir\,d\alpha)$ current-carrying element actually enters into the summation process because,

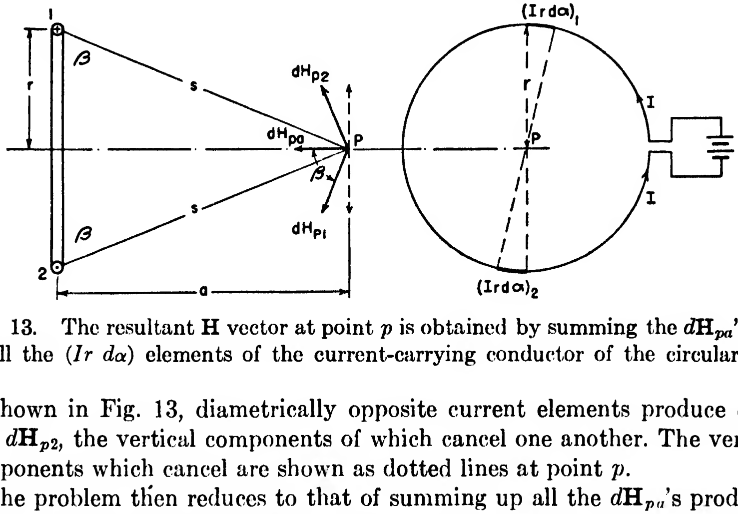


FIG. 13. The resultant **H** vector at point p is obtained by summing the $d\mathbf{H}_{pa}$'s due to all the $(Ir\,d\alpha)$ elements of the current-carrying conductor of the circular coil.

as shown in Fig. 13, diametrically opposite current elements produce $d\mathbf{H}_{p1}$ and $d\mathbf{H}_{p2}$, the vertical components of which cancel one another. The vertical components which cancel are shown as dotted lines at point p.

The problem then reduces to that of summing up all the $d\mathbf{H}_{pa}$'s produced by the various $(Ir\,d\alpha)$ elements around the loop. Since all the $d\mathbf{H}_{pa}$'s are directed along the same line (the central axis of the circular loop) they may be added arithmetically. Using the form shown in equation (12) for dH_p and recognizing that in this case dx of equation (12) is replaceable by $(r\,d\alpha)$ we have

$$dH_{pa} = dH_p \cos\beta = \frac{Ir\,d\alpha}{4\pi s^2}\cos\beta$$

and since $\cos\beta = r/s$,

$$dH_{pa} = \frac{Ir^2}{4\pi s^3}\,d\alpha$$

Inasmuch as α is the only variable and it is measured in the plane of the coil as shown in Fig. 13, the problem of finding H_{pa} is simply that of summing up the $d\alpha$'s around a circle; and in angular measure (that in which $d\alpha$ is expressed) this sum is plainly 2π radians.

Formally,

$$H_{pa} = \frac{Ir^2}{4\pi s^3}\int_0^{2\pi} d\alpha = \frac{Ir^2}{4\pi s^3}\Big[\alpha\Big]_0^{2\pi} = \frac{Ir^2}{2s^3} \quad \text{amp-turns/m} \tag{15}$$

Or, since $s = \sqrt{r^2 + a^2}$ in Fig. 13,

$$H_{pa} = \frac{Ir^2}{2(r^2 + a^2)^{3/2}} \quad \text{amp-turns/m} \tag{16}$$

where a is the distance from the center of the circular turn measured along the central axis of the turn.

If instead of the single turn shown in Fig. 13 we have N turns which are tightly packed together, equations (15) and (16) need only be multiplied by N to obtain H_{pa} which results from the N turns. For example, the magnitude of *H at the center of N tightly packed turns* is

$$H_{\text{center}} = \frac{NI}{2r} \quad \text{amp-turns/m} \tag{17}$$

A tightly packed coil of turns is one in which the cross-sectional area of all the wires that form the coil is contained within radial and axial dimensions which are small relative to the radius of the coil, say 5 per cent.

10. Solenoid Arrangement of Turns. If N turns of a current-carrying conductor are uniformly distributed along a central axis as shown in Fig. 14, the resulting arrangement is called a solenoid. Since the magnetic flux ϕ, which is established across the center cross-section of a long solenoid is very often used as a standard in calibrating ballistic galvanometers to read flux (or, more precisely, change of flux), the method whereby this flux is calculated will be outlined below. As might be anticipated, the center-section flux in which we are chiefly interested (as well as the flux across any other section) is a function only of the dimensions of the solenoid and the current carried by the turns. Since the latter quantities can be measured quite accurately, there is obtained an equally accurate standard of magnetic flux if it is assumed that ϕ can be evaluated as a function of the dimensions of the solenoid and the current.

With N turns distributed uniformly along the axial length of the solenoid as shown in Fig. 14, it is plain that there are N/l turns per unit length. Neglecting the spiral of the actual winding, and assuming that the wire diameter is small compared with the radius r of the solenoid we have $\left(\frac{N}{l}\,da\right)$ turns in the elemental distance da shown in Fig. 14. Distance a is measured along the central axis of the solenoid. In order to be able to use equation (16), we select the origin of a as being at any arbitrary point p along the central axis. In so doing the $\frac{NI}{l}\,da$ mmf

established over the elemental distance da shown in Fig. 14 will manifest itself at point p in the form of an elemental field intensity vector $d\mathbf{H}_{pa}$. In accordance with the derivation of equation (16) we realize that $d\mathbf{H}_{pa}$ is directed along the central axis, and from this equation we also know that the magnitude of $d\mathbf{H}_{pa}$ is

$$dH_{pa} = \frac{(N\,da)Ir^2}{2l(r^2 + a^2)^{3/2}} \tag{18}$$

Now, by letting a become the independent variable (all other factors on the right-hand side of the above equation being constant), we may sum up the effects of all the elemental mmf's, namely, all of the $\left(\frac{NI}{l}\,da\right)$ mmf's which contribute to the establishment of the resultant $\mathbf{H}$ at point p.

The summation process is carried out by integrating with respect to a between the limits of $a = -[(l/2) - d]$ and $a = [(l/2) + d]$ because

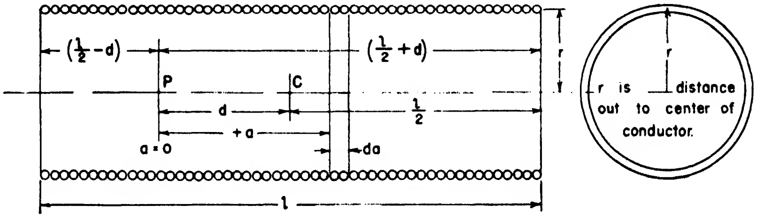


FIG. 14. N-turn solenoid of length l and radius r with linear measure along the axial length reckoned from point p in terms of variable distance a.

it is over this length that elemental mmf's are present and it is over this length that we know that the resultant $\mathbf{H}$ vector is directed along the central axis. The actual details of the summation process are shown below. Starting with equation (18),

$$H_{pa} = \frac{NIr^2}{2l}\int_{-(l/2)+d}^{(l/2)+d}\frac{da}{(r^2 + a^2)^{3/2}}$$

$$= \frac{NIr^2}{2l}\left[\frac{a}{r^2(r^2 + a^2)^{1/2}}\right]_{-(l/2)+d}^{(l/2)+d}$$

$$= \frac{NI}{2l}\left[\frac{\left(\frac{l}{2} + d\right)}{\left[r^2 + \left(\frac{l}{2} + d\right)^2\right]^{1/2}} + \frac{\left(\frac{l}{2} - d\right)}{\left[r^2 + \left(\frac{l}{2} - d\right)\right]^{1/2}}\right] \tag{19}$$

where H_{pa} is expressed in ampere-turns per meter if I is expressed in amperes and the distances in meters.

At the center point, C in Fig. 14, $d = 0$ and equation (19) reduces to

$$H_{a(\text{center})} = \frac{NI}{2\left(r^2 + \frac{l^2}{4}\right)^{1/2}} = \frac{NI}{(4r^2 + l^2)^{1/2}} \tag{19-a}$$

and if $4r^2$ is small relative to l^2 the expression given above reduces simply to

$$H_{a(\text{center})} = \frac{NI}{l} \quad \text{amp-turns/m} \tag{20}$$

At the ends of the solenoid where $d = l/2$, equation (19) reduces to

$$H_{a(\text{ends})} = \frac{NI}{2(r^2 + l^2)^{1/2}} \tag{21}$$

and if r^2 is small compared to l^2

$$H_{a(\text{ends})} = \frac{NI}{2l} \quad \text{amp-turns/m} \tag{22}$$

It will be observed that the end values of H_a (the field intensities along the central axis) are precisely one-half the center-section value of H_a in the case of the long solenoid.

Example. The magnetic field intensity vector **H** which crosses the mid- or center section of a *long* solenoid is essentially constant over the entire face of this section (as may be proved either experimentally or by some involved calculations). It will be shown in Sections 11 and 12 that the magnetic flux, ϕ crossing the center section of the solenoid may be calculated as

$$\phi_{(\text{center})} = \mu HA = \frac{\mu NI(\pi r^2)}{l} \quad \text{webers} \tag{23}$$

where $\mu = \mu_0 = 4\pi \times 10^{-7}$ since the solenoid is presumed to have an air core
N is the total number of turns
I is the current in amperes
r is the radius out to the center of the conductor, as shown in Fig. 14, expressed in meters
l is the axial length of the solenoid expressed in meters.

Equation (23) is used in connection with the long-solenoid method of calibrating ballistic galvanometers.

11. Definition of the B Vector. Thus far our attention has been directed almost exclusively to the evaluation of the magnetic field intensity vector **H**. No particular significance has been attached to the **H** vector aside from the fact that $H\,\Delta l$ produces a magnetic potential drop. The real value of the magnetic intensity vector **H** from a practical

point of view is that **H** can be used to define, and evaluate directly, a most important physical quantity called the *magnetic flux density* vector and symbolized by **B**. This **B** vector is defined as

$$\mathbf{B} = \mu\mathbf{H} = \mu_0\mu_r\mathbf{H} \quad \text{webers/sq m} \quad \text{(in mks units)} \tag{24}$$

where **H** is expressed in ampere-turns per meter if rationalized mks units are used
$\mu_0 = 4\pi \times 10^{-7}$ is the permeability of free space or air
μ_r is the relative permeability of the medium in which **H** and **B** act.

Wherever an **H** vector exists, a **B** vector also exists and, provided μ_r and **H** can be evaluated, the value of **B** follows immediately from equation (24).

12. Magnetic Flux ϕ. It has been shown that **H** is a space-distributed vector which has as its origin current loops (NI); hence by equation (24) it follows that magnetic flux density is a space-distributed vector having the same origin. Since **B** is selected by definition to be the *magnetic flux density* vector, the product of B and the area A which is pierced at right angles by the **B** vector will by definition be the *magnetic flux* which crosses area A. Magnetic flux is measured in webers in mks units and is symbolized by ϕ.[3] Thus the amount of magnetic flux ($\Delta\phi$) which crosses any increment of area ΔA is

$$\Delta\phi_{(\text{magnetic flux})} = \mathbf{B} \cdot \mathbf{A} = B \cos \theta \big]_{\mathbf{B}}^{\Delta\mathbf{A}} \Delta A \quad \text{webers} \quad \text{(in mks units)} \tag{25}$$

The Δ's simply imply that the quantities following are specified increments. The significance of the above equation is illustrated in Fig. 15 where $B_{\text{av.}}$ represents the space-averaged value of B over the small patch of area ΔA. The angle θ is the angle between the **B** vector and the area $\Delta\mathbf{A}$ considered as a vector. (It will be remembered that the direction of a vector area is normal to the face of the area.) After multiplying the component of **B** which is normal to the face of the area by the area we obtain the magnetic flux that pierces or crosses the area.

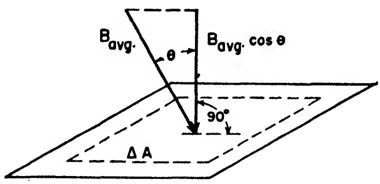


FIG. 15. The flux crossing area ΔA is the normal component of the space-averaged value of flux density over ΔA times ΔA.

[3] Another commonly used unit of magnetic flux is the cgs line or maxwell. As will be shown later the maxwell, as a unit of measure, is 10^8 times smaller than the weber. Actually the weber is too large a unit of measure of magnetic flux for most practical purposes, and the maxwell is too small a unit, as will become more evident as we proceed.

In the following chapter we shall consider in some detail the physical meaning of magnetic flux and magnetic flux density. In the examples which are now given to illustrate the use of equation (25) we shall simply consider magnetic flux as the *projected effect* of some current-carrying loops which are in the neighborhood. The analogy between *magnetic flux* considered as the *projected effect of moving charges* and *electric flux* considered as the *projected effect of stationary charges* may be helpful from a conceptual point of view.

The evaluation of the space-averaged value of the **B** vector over some specified area is encountered in many problems which require a knowledge of the amount of magnetic flux which crosses the area. Usually the **B** vector varies at the most in only one direction, that is, along one dimension of the specified area. If we can find the space-averaged value of B along this dimension, then plainly this averaged value represents the average value of B over the entire area.

Formally, we find the space-averaged value of B, say along the x direction, to be

$$B_{\text{av.}} = \frac{1}{x_2 - x_1} \int_{x_1}^{x_2} B_x \, dx \tag{26}$$

where x_2 and x_1 are constants which are defined by the end points of the line along the x axis which bounds the specified area and along which we presumably know B_x as a function of the variable x.

The reader should appreciate the fact that, after the operations which are indicated in equation (26) are performed, the value of $B_{\text{av.}}$ is the same, to ordinary engineering accuracy, as that which might be obtained by either of the two following methods.

(1) Take several equally spaced values of the variable x and evaluate B_x at each of the selected values of x; then divide the sum of the B_x's found by the number of B_x's to obtain $B_{\text{av.}}$. (In certain cases some judgment must be exercised in choosing the number and placement of these equally spaced values of x.)

(2) Plot a curve of B_x versus x from the relationship which expresses B_x as a function of x. Measure the area under this curve in terms of squares. Then, since one side represents so many webers per square meter and the other side so many meters, depending upon the scale employed in plotting, the area may be readily converted into webers per meter which, if divided by the base $x_2 - x_1$ expressed in meters, yields the space-averaged value of B along the line from $x = x_1$ to $x = x_2$.

Method (2) is a graphical method of performing the operations indicated in equation (26). This brief reminder concerning the evaluation of space-averaged values is given in order that the reader will treat the

integral signs employed in the following examples for what they are; simply short methods of finding $B_{av.}$ which, to the accuracy normally required, could have been found by less formal methods. The integrations should not detract the reader's attention from the problem at hand, namely, the calculation of magnetic flux crossing a surface, since the integrations can always be ignored and the indicated operations performed otherwise.

Example 1. Let it be required to find the magnetic flux which crosses the $a \times b$ area in Fig. 16, expressed as a function of $B_{max.}$.

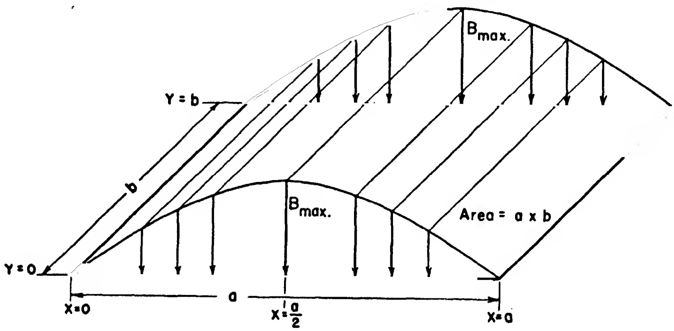


FIG. 16. Sinusoidal space distribution of **B** along the x axis.

Since the **B** vectors are normal to the face of the area in question, the problem reduces to finding the space-averaged value of B over this area. The variation of B along the a dimension of the area may in this case be written as

$$B_x = B_{max.} \sin\left(\frac{\pi}{a} x\right)$$

where B_x is the magnitude of the flux density vector at any point x from the $x = 0$ origin

$B_{max.}$ is the maximum magnitude of the flux density at $x = a/2$.

The factor $\sin(\pi x/a)$ is simply an analytical expression which allows B_x to be equal to $B_{max.}$ at $x = a/2$, and to be zero at $x = 0$ and at $x = a$, and at the same time describe the sinusoidal space variation of B_x which is specified in Fig. 16.

The average value of a sinusoidal variation over one-half cycle of the variation is either known to be $2/\pi$ times the maximum value or else evaluated as

$$B_{x\,av.} = \frac{1}{a}\int_0^a B_{max.} \sin\left(\frac{\pi}{a} x\right) dx = \frac{2}{\pi} B_{max.}$$

Hence

$$\phi_{a \times b} = B_{av.}A = 0.637 B_{max.}(a \times b)$$

Example 2. *Magnetic Flux Inside a Coaxial Cable.* Let attention be focussed on the coaxial arrangement of current-carrying conductors shown in cross-section in Fig. 10, page 208. The first step in finding the magnetic flux which circulates (or exists) in the region between the two conductors ($r_i < x < r$ in Fig. 10) is to visualize a well-defined area which is pierced by the flux. *The visualization of areas pierced by magnetic flux is usually the most important step in magnetic field problems.*

Let it be required to find the magnetic flux that pierces the rectangular area in Fig. 10 which is bounded by (1) the x axis between $x = r$ and $x = r_i$; (2) 3 km of axial length, which is directed normal to the plane of the page in Fig. 10 under the following specified conditions:

$$r = 0.5 \text{ cm} \quad r_i = 2.5 \text{ cm} \quad I = 50 \text{ amp} \quad \mu_r \text{ of medium} = 1$$

The magnitude of **H** which pierces the specified area is known to be

$$H = \frac{I}{2\pi x} \quad \text{amp-turns/m} \qquad (r < x < r_i)$$

and also from previous considerations we know that H does not vary along the axial length of the cable provided we keep about 10 or 15 cm (in this case) from the very ends of the line.

Thus the problem reduces essentially to that of finding the space-averaged value of B along the x direction of the area in question. As a function of x, B may be written

$$B = \mu_0 H = \mu_0 \frac{I}{2\pi x} = 4\pi \times 10^{-7} \frac{50}{2\pi x} = \frac{10^{-5}}{x} \quad \text{weber/sq m}$$

Then

$$B_{\text{av.}} = \frac{1}{(r_i - r)} \int_r^{r_i} B\, dx = \frac{10^{-5}}{(0.025 - 0.005)} \int_{0.005}^{0.025} \frac{dx}{x}$$

or

$$B_{\text{av.}} = 0.5 \times 10^{-3} \left(\ln \frac{0.025}{0.005} \right) = 0.5 \times 10^{-3}(1.61)$$

$$= 0.805 \times 10^{-3} \quad \text{weber/sq m}$$

The magnetic flux which pierces the specified area is then

$$\phi = B_{\text{av.}}(A) = 0.805 \times 10^{-3}(0.02 \times 3000) = 0.0483 \quad \text{weber}$$

Calculations of this kind are widely used in inductance calculations which are considered later.

Example 3. *Magnetic Flux between Two Parallel Wires.* Refer to Fig. 9, page 207. Let it be required to find the magnetic flux that pierces the area which is bounded by (1) the straight line joining the surfaces of the conductors, that is, from $x = r$ to $x = D - r$; (2) 3 km of axial length of this parallel-wire transmission line under the following specified conditions:

$$D = 0.50 \text{ m} \quad r = 0.005 \text{ m} \quad I = 100 \text{ amp} \quad \mu_r \text{ of medium} = 1$$

The problem may be solved in either of two ways; by applying the principle of superposition to the flux established by each wire separately, or by making use of the expression which has been derived for the resultant value of **H** in the region over which we desire to know the magnetic flux. [See equation (8).] Of the two methods, the first is much the easier from a mathematical point of view, but in this case we choose to use the expression in equation (8) and let the reader check the result by the principle of superposition.

Since in equation (8) $y = D - x$,

$$H_{\text{res.}} = \frac{I(x + D - x)}{2\pi x(D - x)} = \frac{ID}{2\pi(Dx - x^2)} \quad \text{amp-turns/m}$$

$$B_{\text{res.}} = \mu_0 H_{\text{res.}} = \frac{(4\pi \times 10^{-7})(100)(0.5)}{2\pi(0.5x - x^2)} = 10^{-5}\frac{1}{(0.5x - x^2)}$$

The space-averaged value of $B_{\text{res.}}$ is obtained in the usual manner:

$$B_{\text{av.}} = \frac{10^{-5}}{0.49}\int_{0.005}^{0.495} \frac{dx}{(0.5x - x^2)} = \frac{10^{-5}}{0.49}\left[-\frac{1}{0.5}\ln\frac{0.5 - x}{x}\right]_{0.005}^{0.495}$$

If the reader is not familiar with this particular integral he may readily check it if he knows that $(d/dx)(\ln y) = (1/y)(dy/dx)$ where y is a function of x, which in this case we shall take as $(0.5 - x)/x$. Then

$$\frac{d}{dx}\left[\ln\left(\frac{0.5 - x}{x}\right)\right] = \frac{1}{\dfrac{(0.5 - x)}{x}}\left(\frac{-x - 0.5 + x}{x^2}\right) = \frac{-0.5}{(0.5x - x^2)}$$

which, when the outside coefficient of $(-1/0.5)$ is considered, shows that the expression which was taken for the integral can be differentiated to equal the expression under the integral sign. This is the test for all integrals.

After substituting in the limits in the expression for $B_{\text{av.}}$ we find that $B_{\text{av.}} = 37.5 \times 10^{-5}$ weber/sq m and, since the area involved is (3000×0.49) sq m, *the magnetic flux is* 0.551 weber to slide rule accuracy.

13. Continuity of Magnetic Flux. The fact that magnetic lines (whether they be the **H** vectors or the **B** vectors) close on themselves follows from the fact that the theory of magnetism is based on the circuital law. See equations (5) and (6). The circuital law of magnetism demands that the **H** or **B** lines be linked to the current loop as the neighboring elements of a log chain are linked.

This concept of magnetism leads to the conclusion that over any closed surface there is as much inwardly directed flux as there is outwardly directed flux. If, for example, we consider the region which is bounded by the six flat surfaces shown in Fig. 17 we note that any flux density line like the α line which crosses surface $o'a'b'c'$ outwardly will, to complete its circuit, have to cross some other surface, say surface $oabc$, inwardly.

Whether the α line has its origin (that is, crosses the face of a current loop) inside the region or outside the region makes no difference. If a flux density line like the β line has its origin within the region and its circuit never takes it outside the region, it represents neither outwardly directed flux density nor inwardly directed flux density since it never crosses the closed surface which bounds the region.

These facts may be written symbolically:

$$\Sigma B_A \cos \theta \Big]_{\mathbf{B}_A}^{\Delta \mathbf{A}} \Delta A = 0 \quad \text{(over a closed surface)} \tag{27}$$

This equation merely states that the net outwardly directed magnetic flux over a closed surface is zero. This concept is of little use in an

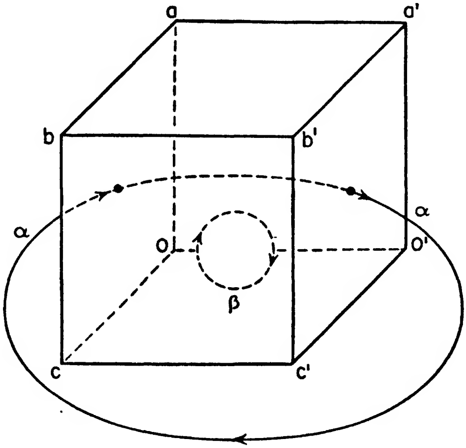


FIG. 17. Magnetic flux *outwardly* directed over a closed surface equals zero.

elementary course but it is of importance in wave propagation studies. It is presented here merely to show how the circuital nature of the magnetic flux density vectors demands that there be no termination point or area for magnetic flux. This concept contrasts sharply with the corresponding situation in the electric field:

$$\Sigma D_A \cos \theta \Big]_{\mathbf{D}_A}^{\Delta \mathbf{A}} \Delta A = \psi = Q \quad \text{(over a closed surface)} \tag{28}$$

where Q is the net or surplus electric charge within the interior of the closed surface. This will be remembered as Gauss' theorem and as a very useful device in finding the distribution of the electric flux lines in the vicinity of symmetrical conductor arrangements. The corresponding equation in magnetic field theory, equation (27), is not nearly so helpful since the current loops which are the sources of the magnetic flux do not

enter this equation; they are only *linked* to it by way of the circuital law of magnetism.

14. Summary. The important facts which have been presented thus far in the study of the magnetic field are:

(1) The **H** vector can, at least formally, be evaluated in terms of the current loops (NI) to which the closed **H** loop is linked.

(2) The flux density vector **B** follows directly from the product of μ and **H**.

(3) The magnetic flux crossing any specified area **A** is evaluated as $\mathbf{B}_{\text{av.}} \cdot \mathbf{A} = B_{\text{av.}} \cos \theta]_{\mathbf{B}}^{\mathbf{A}} A$.

The fact that the permeability of free space, μ_0, is dimensionally equal to fI^{-2}, where f is force and I is current, forms the basis for two important applications of magnetic field theory which are considered in the next two chapters.

PROBLEMS

All results in this set of problems are to be expressed in rationalized mks units.

1. What is the "pole strength" of each of two like magnetic poles if at a distance of separation of 2 cm in free space (or air) they exhibit mutual forces of repulsion of 0.9 newton?

Note: Pole strengths are expressed simply in terms of unit pole strengths of the particular system of units employed; for example, if rationalized mks units are employed in connection with equation (1) the pole strengths are expressed in *rationalized mks pole strengths.*

2. What is the current equivalent around a circular path of 1 m radius (on a time averaged basis) if a charged particle of +0.2 coulomb is circulating around this path at a uniform speed of 10 rps?

3. On the basis of 1 amp times 1 m (in a closed loop) being the equivalent of a unit magnetic pole, what is the magnetic pole strength equivalent of the circulating charge described in Prob. 2?

4. What magnetic potential rise is developed by a 10-turn coil which carries a current of 5 amp?

5. What is the magnetic potential drop around any closed path which links with the 10-turn coil of Prob. 4?

6. A round wire 0.4 cm in radius has a current density of 200 amp sq cm. What is the magnitude of the magnetic field intensity **H** around the circular periphery of the wire?

Note: Owing to symmetry, the magnitude of **H** is constant at all points along the circular path.

7. Two parallel rectangular surfaces (each 0.8 m x 0.5 m in area) are separated from one another by a distance of 4 mm in air and are maintained at a magnetic

potential difference of 4.0 amp-turns by current loops which surround the surfaces. Find the magnetic potential gradient in the region between these two surfaces.

8. In Fig. 18 are shown (in cross-section) seven long straight conductors. Each conductor carries 2 amp directed out of the plane of the page as indicated by the • symbols. What magnetomotive force is linked by the circular *abca* path? By the irregular *abdca* path?

9. What is the numerical value of H_r in Fig. 18 at point a if I/cond. is 2 amp and $r = 1/\pi$ m?

10. Employing equipotential surfaces as shown in Fig. 6, page 204, show that the Hl magnetic potential drop around the *abdca* path in Fig. 18 is the same as the magnetic potential drop around the circular path.

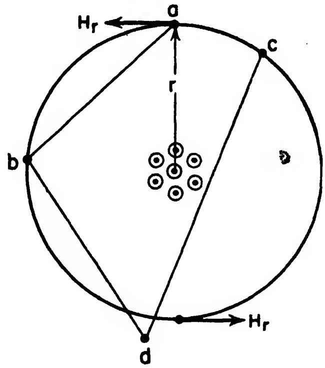


FIG. 18. Cross-sectional view of seven long straight conductors. See Probs. 8, 9, 10, and 11.

11. Assume that the cross-section of seven conductors shown in Fig. 18 is that of a huge square coil of wires, the cross-section shown being near the center of one side. The side of this coil which is parallel to the one shown in cross-section is directly above the cross-section shown in Fig. 18. As in Prob. 9, I/cond. = 2 amp and $r = 1/\pi$ m. Draw the equivalent magnetic circuit of the $2\pi r H_r$ circular path which shows the mmf rise at the physical place in the circuit where the path crosses the face of the current loops. Designate on this equivalent circuit the numerical value of the mmf as well as the polarity of this mmf; − representing the south pole face of a current loop and + the north pole face of a current loop. The total mmf may be represented as a single source.

12. The inner conductor of the coaxial arrangement shown in Fig. 10, page 208, is assumed to be carrying 25 amp directed out of the plane of the page, and the outer conductor 25 amp directed inwardly.

$$r = 0.5 \text{ cm} \quad r_i = 2.5 \text{ cm} \quad r_0 = 2.6 \text{ cm}$$

(a) Find H_1 at $x = x_1 = 0.1$ cm.
(b) Find H_2 at $x = x_2 = 1.0$ cm.
(c) Find H_3 at $x = x_3 = 2.55$ cm.
(d) Find H_3 at $x = x_3 = 2.60$ cm.

13. The parallel wires shown in Fig. 9, page 207, are assumed to be carrying 100 amp directed as shown. The radius r of each conductor is 0.5 cm, and the *c*-to-*c* separation of these conductors (D) is 0.50 m.

(a) Which is the north pole face of the loop formed by these wires and the end-connecting circuits?

(b) What mmf is produced by this loop?

(c) Make a plot of the resultant magnitude of the **H** vector (along the straight line joining the conductor surfaces) versus x (the distance from the center of the left-hand conductor).

Note: The mmf of the loop itself is not included unless an equivalent circuit is being drawn, even though the line here selected coincides with the face of the current loop. The plot in question is to extend from $x = 0.005$ m to $x = 0.495$ m.

(*d*) What is the space-averaged value of H along the x axis from $x = 0.005$ m to $x = 0.495$ m determined by the graphical method which divides the area under the curve by the base line of the curve?

14. What is the magnitude and direction of the $\mathbf{H}$ vector which exists at a point 0.50 m directly above the center of the left-hand conductor shown in Fig. 9, page 207, if $I = 100$ amp, $r = 0.005$ m, and $D = 0.50$ m?

Note: Employ the principle of superposition and specify the direction of the resultant $\mathbf{H}$ vector relative to the $+x$-axis direction using positive (or counterclockwise) angular measure from this reference. Compare the *general* direction of the resultant $\mathbf{H}$ vector thus obtained with the direction of the circles which enclose the left-hand conductor of Fig. 21, page 187, considering that this conductor is carrying current directed out of the page, and selecting a point which is directly above the conductor by about the distance of separation of the conductors. This comparison is necessarily only qualitative.

15. Determine the magnitude and direction of the $\mathbf{H}$ vector in Fig. 9, page 207, due to both line conductors at a point which is at $x = -0.50$ m (that is, to the left of the left-hand conductor) if

$$I = 100 \text{ amp} \quad r = 0.005 \text{ m} \quad D = 0.50 \text{ m}$$

16. Find the magnetic field intensity H at the center of a tightly packed coil of 20 circular turns if the effective diameter of this coil is 10 in. and the current is 150 ma.

17. Derive the expression for the resultant magnetic field intensity H at the center of one square-shaped turn which carries I amp and has a single-side length of s m.

18. Refer to Fig. 13, page 212, and find the summation of all the $H\,dl$'s along the central axis from a point which is 10 m to the left of the cross-sectional view to a point which is 10 m to the right of the center of the coil; assume that the single-turn coil carries I amp in the direction indicated by the + and · symbols and that the radius of the circular turn is 0.1 m.

Note: This problem may be solved either graphically or analytically, and in either case an answer which is accurate to within 2 per cent is satisfactory; or the answer may, in view of the numbers specified and the accuracy desired, be written down from inspection. To check the latter type of answer one may note that, for the variable a measured from left to right,

$$d(H_{pa}\,da) = \frac{-Ir^2\,da}{2(r^2 + a^2)^{3/2}}$$

$$\Sigma(H_{pa}\,da) = -\frac{I}{2}\int_{-10}^{+10} \frac{r^2\,da}{(r^2 + a^2)^{3/2}} = -\frac{I}{2}\left[\frac{a}{(r^2 + a^2)^{1/2}}\right]_{-10}^{+10} \quad \text{amp-turns}$$

19. A long distributed single-layer coil of wire of 1000 turns is measured and found to have the following dimensions:

$$\text{axial length } l = 1.80 \text{ m} \quad \text{radius of circular turns } r = 0.05 \text{ m}$$

From equation (19) it is known that the magnitude of $\mathbf{H}$ directed along the central axis of the coil is the following function of d, where d is distance measured along the

central axis (in either direction) from the point which is midway between the ends of the coil:

$$H_{pd} = \frac{NI}{2l}\left[\frac{(0.5l + d)}{[r^2 + (0.5l + d)^2]^{1/2}} + \frac{(0.5l - d)}{[r^2 + (0.5l - d)^2]^{1/2}}\right] \quad \text{at/m}$$

(*a*) Evaluate H_{pd} for $d = 0$, $d = 0.3l$, $d = 0.4l$, and for $d = 0.5l$; for I amp.

(*b*) Plot a curve of H_{pd} over the entire length of the coil, that is, from $d = 0.5l$ to $d = -0.5l$ versus d.

(*c*) Determine the magnetic potential drop from $d = 0.5l$ to $d = -0.5l$ from the area under the curve plotted in part (*b*).

20. Determine the magnitude and direction of the **H** vector at points p_c and p_o which both lie in the plane of the rectangular coil shown in Fig. 19 to a reasonable degree of accuracy by neglecting the effects of the short sides. Point p_c is at the center of the coil and point p_o is 11 cm from the center of the coil.

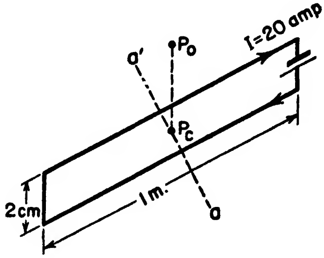


FIG. 19. See Prob. 20.

21. What is the flux density at the center of a tightly packed coil of 20 circular turns if the effective diameter of this coil is 25 cm and the current is 150 ma? Relative permeability, $\mu_r = 1$.

22. What is the flux density at the center of one square-shaped turn which carries 20 amp and which has a single-side length of 30 cm? ($\mu_r = 1$.)

23. What is the flux density at a point midway between the two line conductors shown in Fig. 9 and on the straight line joining the surfaces of the conductors if $I = 100$ amp, $D = 0.50$ m, and $r = 0.005$ m? (The relative permeability of the medium is unity.)

24. Consider a rectangular area of 10 cm by 50 cm which is near a long straight wire that is carrying 100 amp. The current-carrying wire lies in the plane of the rectangular area, and the 50 cm dimension of the area is parallel to this wire. The distance from the wire center to the nearest 50 cm side of the area in question is 8 cm. Find the magnetic flux crossing the 500-sq-cm area.

25. What is the flux density at a point, expressed in webers per square meter, if it is known that the flux density at this point is 5000 maxwells/sq cm? It is further known that 1 weber $\equiv 10^8$ maxwells or (No. of) webers/maxwell is 10^{-8}.

26. What is the quantity of magnetic flux which crosses the mid-cross-sectional area of the 1000-turn coil described in Prob. 19 if it is known that the **H** vector is uniformly distributed over this entire area and that, as a very good approximation, the value of H is NI/l amp-turns/m at the mid-section? In this case, $I = 4$ amp, and $\mu_r = 1$.

27. The two sides of the rectangular area shown in Fig. 16, page 218, are: $a = 21$ cm, $b = 25$ cm. At $x = 7$ cm (from $x = 0$ in Fig. 16) the flux density has a known value of 0.00866 weber/sq m. As indicated, the flux density along the x direction has a sinusoidal space distribution between $x = 0$ and $x = 21$ cm (in this problem). The flux density does not vary along the y direction. What amount of magnetic flux crosses the $a \times b$ area?

28. Refer to Fig. 10, page 208, and consider that we are dealing with a 3-km length of this coaxial line, the conductors of which are presumably copper. The current per conductor is 25 amp. Determine the total flux that pierces or crosses the rectangular area which is bounded by: (1) the x axis between $x = 0$ and $x = r_i$; (2) 3 km of axial length.

$$r = 0.5 \text{ cm} \quad r_i = 2.5 \text{ cm} \quad r_o = 2.6 \text{ cm} \quad \mu_r = 1 \quad \text{(throughout)}$$

The relative permeability of copper is essentially unity.

29. Repeat Prob. 28 for the area which is bounded by: (1) the x axis between $x = 0$ and $x = r_o$; (2) 3 km of axial length.

30. Refer to Fig. 9, page 207, and consider that we are dealing with a 3-km length of this parallel-wire line, the conductors of which are presumably copper. Determine the total flux that crosses the rectangular area which is bounded by: (1) the x axis between $x = r$ and $x = (D - r)$; (2) 3 km of axial length of line.

$$D = 0.50 \text{ m} \quad r = 0.005 \text{ m} \quad I = 100 \text{ amp} \quad \mu_r = 1 \quad \text{(throughout)}$$

Note: It is expected that the student solve this problem by applying the principle of superposition. See page 220.

31. Repeat Prob. 30 except for the area, which is here considered to be bounded by: (1) the x axis between $x = 0$ and $x = D$; (2) 3 km of axial length of line.

CHAPTER IX

Magnetic Forces

1. Dimensional Prediction of Magnetic Forces. Since magnetic field theory as outlined in Chapter VIII was originally based on forces which were known to exist between current-carrying loops, it is to be expected that this theory should be capable of accounting for forces which are developed on current-carrying conductors. The magnetic flux density vector **B** which was given a somewhat arbitrary unit of measure in Chapter VIII now takes on added significance both from the point of view of the units employed in expressing it and from the point of view of its dimensional characteristics.

The flux density vector may be expressed dimensionally as

$$B = \mu_0^{\,1} H^1 \tag{1}$$

since μ_r in equation (24) on page 216 is a dimensionless numeric, as will be shown more clearly in Chapter XI. From equation (1-a), page 198, we know that from a dimensional point of view

$$\mu_0 = f^1 I^{-2} \tag{2}$$

and from equation (4), page 202, we know from this definition of H that dimensionally

$$H = I^1 l^{-1} \tag{3}$$

Hence the dimensional expression for B becomes

$$B = (f^1 I^{-2})(I^1 l^{-1}) = f^1 I^{-1} l^{-1} \tag{4}$$

or since $I^{-1} = Q^{-1} t^1$

$$B = f^1 Q^{-1} t^1 l^{-1} = f^1 Q^{-1} v^{-1} \tag{5}$$

where f represents force and v velocity, as should be evident.

An examination of equations (4) and (5) will show that magnetic flux density is a physical quantity which, however arbitrarily defined, expresses: (1) the force per unit current of unit length, that is, per unit (Il), or (2) the force per unit charge of unit velocity, that is, per unit

(Qv). There is no fundamental difference between these two concepts since the second was derived directly from the first; for example, a unit charge possessing unit velocity directed around a loop of unit length would pass any point of the loop once each second and, since $I = Q/t$, this moving charge represents unit current over a unit length of circuit. With respect to any particular section of the circuit, the current in question is, of course, a time-averaged value of current.

On the basis of this dimensional investigation, the dimensional forms shown in equations (4) and (5) may be arranged as algebraic equations to read

$$f = k_1 BIl \tag{6}$$

$$f = k_2 BQv \tag{7}$$

and if our units of B have been selected properly, as they have in a systematic set of units, the proportionality constants, k_1 and k_2, reduce to unity. Then, in systematic units, equations (6) and (7) become

$$f = BIl \tag{8}$$

$$f = BQv \tag{9}$$

where the force f is given in newtons if B is expressed in webers per square meter, I in amperes, l in meters, Q in coulombs, and v in meters per second. Other systematic units may also be employed in these equations.

The dimensional investigation which was employed in arriving at equations (8) and (9) has been given largely to let the reader appreciate what might be expected from the physical quantity B which entered magnetic field theory simply by way of definition. The reader should, however, accept equations (8) and (9) as basic laws of nature, for example, as he accepts the law of gravitation. It must be realized that magnetic field theory, like any other theory, is devised for the purpose of tying together physically observed phenomena which, without the aid of a consistent theory, would appear as so many isolated facts.

The electrical engineer employs equations (8) and (9) to account quantitatively for a wide variety of phenomena; from motor action in huge electromagnetic machinery to the deflection of a fine stream of electrons in a magnetic-deflection cathode-ray oscillograph. The remainder of this chapter is devoted to the interpretation of these equations under various conditions encountered in engineering practice.

Since the dimensional derivation employed in this section does not take into account the directional properties of the vectors $\mathbf{B}$, $\mathbf{l}$, and $\mathbf{v}$, it is to be expected that the simple forms of the basic laws given in equa-

tions (8) and (9) will have to be examined more carefully and in the light of the relative orientations or directions of the vector quantities involved.

2. Right-Angle Relationships between Vectors (Il), B, and f. Although its directional properties have not as yet been considered, it is plain that force **f**, in equations (8) and (9), is a vector quantity which will be defined in some manner by the vectors (Il) and **B** in one case and by **v** and **B** in the other.

The manner in which the **f** direction is defined by the relative directions of (Il) and **B** is shown in Fig. 1. If the (Il) element is directed normally

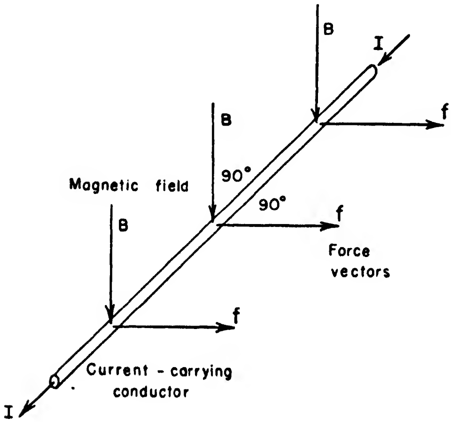


FIG. 1. Magnetic forces are directed at right angles to the plane in which **B** and the current-carrying conductor are located.

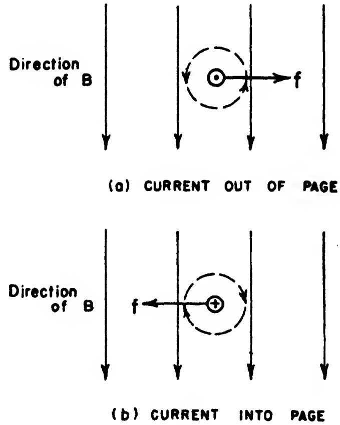


FIG. 2. The direction of magnetic forces.

to the direction of the flux density vector as illustrated, the force developed on (or in) the (Il) element is directed at right angles to the plane in which (Il) and **B** are located. In this case, equation (8) applies directly, but for completeness we add, mentally, the direction of the force as it appears in nature.

The statement that "the force **f** is directed at right angles to the plane in which (Il) and **B** are located" needs further qualification in order that the correct direction along the normal be clearly understood. Various rules have been devised to specify this direction; all are artifices designed to help us remember the direction known, from experiments, to be correct. One of the simplest procedures to follow, particularly where the conductors are shown in cross-section as they are in Fig. 2, follows: (1) Think of grasping the conductor with the right hand so that the thumb points along the axial length of the conductor in the direction of

positive current ($+Il$). The fingers will then encircle the conductor in the positive direction of the magnetic field which is established by the current-carrying conductor itself. (2) Locate the side of the conductor on which the encircling flux (established by the conductor) opposes the original flux (the flux that exists at this point without the current-carrying element present). (3) The force is then directed toward the region where the encircling flux directly opposes the original flux.

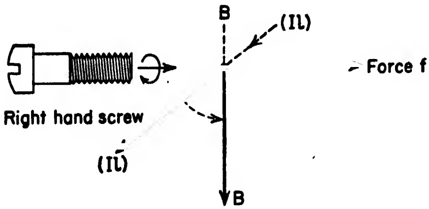


FIG. 3. Think of turning (Il) to **B** position to obtain direction of force.

The examples given in Fig. 2 should help to clarify the application of this method, the application being much simpler than the verbiage indicates. In connection with the application of this rule for finding the direction of force, it will be remembered that the symbol + indicates that the vector quantity considered is directed into the plane of the page and that the symbol · indicates that the quantity is directed out of the page.

A more concise rule than the one given above is illustrated in Fig. 3. It is known as the right-hand screw rule. Here the slot in the head of a right-hand screw is first thought of as coinciding in direction with the direction of the (Il) element. Then, the slot is turned (through the smaller angle) to the **B** direction, and the right-hand screw travels in the force direction. Since this rule is so easily applied to any point in space and to moving charges with equal facility, it is widely used.

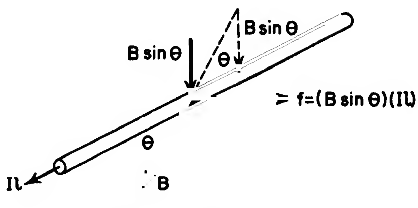


FIG. 4. Force is directed normal to the plane in which **B** and (Il) lie but is reduced in magnitude over that shown in Fig. 1.

If, in the plane in which (Il) and **B** are located, **B** is not directed normally to the direction of (Il), B in equation (8) must be replaced with the component of the total **B** vector which is normal to (Il) considered as a vector. As illustrated in Fig. 4, the component of the total **B** vector which is normal to the direction of the current-carrying element (Il) is $B \sin \theta]_{\mathbf{B}}^{(Il)}$. We may either write equation (8) in the form

$$f = B \sin \theta]_{\mathbf{B}}^{(Il)}\ (Il) \tag{8-a}$$

or remember that the value of B which appears in equation (8) is the component of the actual **B** which is normal to the direction of the current-carrying conductor.

The component of B which is parallel to the current-carrying conductor ($B \cos \theta$) does not develop force on the conductor. If, for example, the conductor shown in Fig. 1 were turned through 90° so that the direction of (Il) coincided with the direction of **B** the magnetic forces shown in Fig. 1 would be reduced to zero. In most practical situations, a maximum force is desired from a given **B** and a given (Il), and the conductor is placed with its axial length at right angles to the direction of the magnetic field as illustrated in Figs. 1 and 2.

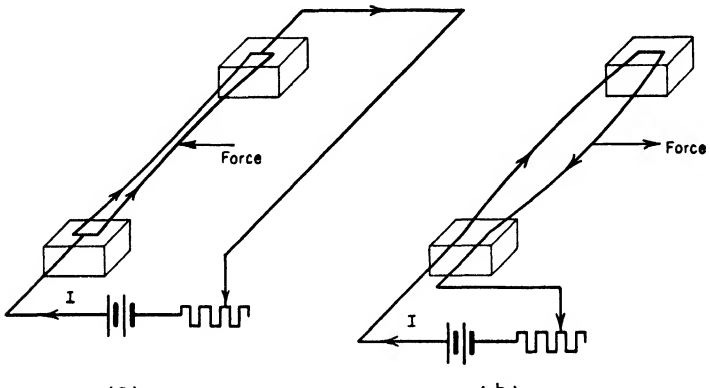


FIG. 5. Adjacent conductors carrying currents in the same direction experience mutual forces of attraction; if carrying oppositely directed currents these conductors experience mutual forces of repulsion.

Example. Refer to Fig. 5-a, and consider a 2-m section of the parallel current-carrying conductors which is midway between the supports. It will be assumed that the conductors are sufficiently long so that each wire taken separately will produce a circular magnetic field about its axis which can be evaluated as

$$B = \mu_0 \mu_r H$$

where $H = I/2\pi r$ amp-turns/m. It will be assumed further that the current density is uniformly distributed over the cross-sectional area of each conductor or else that the distance of separation which we employ has been so chosen as to account for the fact that moving charges experience forces which, in this case, make for higher current densities in the portions of the conductors which face one another. Stated more briefly, we propose to neglect *proximity effect,* which in this case would make for higher forces than we shall obtain in our calculations.

Let $I = 500$ amp, and let the center-to-center separation of the conductors be 1 cm. Then, if the effect of the local field established by the conductor itself is neglected, the magnetic flux density at the center of this conductor is

$$B = \mu_0\mu_r H = (4\pi \times 10^{-7})(1)\left(\frac{500}{2\pi \times 0.01}\right) = 0.01 \quad \text{weber/sq m}$$

and the force on a 2-m section of each conductor is

$$f = BIl = (0.01)(500)(2) = 10 \text{ newtons} \quad \text{(or 2.25 lb)}$$

It is suggested that the reader draw a cross-sectional view of the conductors shown in Fig. 5-a and apply the rule illustrated in Fig. 2; not only for the practice of using that rule but also to see that the directions of **B** and (Il), the magnitudes of which are used in the above force equation, are directed normally to each other.

The *observer* method which is described in Chapter VIII may also be used for finding the direction of the magnetic forces between two current-carrying conductors. To apply this method to Fig. 5-a, for example, the observer places himself between the two loops (which in this case have a common return path) by placing himself between the two parallel conductors. If he looks to the right he sees what appears, to his limited range of vision, to be *counterclockwise*-directed current, whereas if he looks to the left he sees *clockwise*-directed current. He decides, therefore, that the two loops or at least the portions of the loops that he observes will be attracted toward one another as would a north pole and a south pole of two permanent magnets.

3. Right-Angle Relationships between Vectors v, B, and f. If it is recognized that

$$Q\mathbf{v} = (It)\left(\frac{l}{t}\right) = Il \tag{10}$$

then any of the rules for finding the directions of forces on current-carrying conductors apply equally well to forces on moving charges. The symbol ⊙ in Fig. 2, for example, may be interpreted as a positive charge directed out of the page and, since this moving charge constitutes a current, it sets up its own localized magnetic field (shown by the dashed circle) in precisely the same manner as the moving charges which constitute current flow within a conductor.

In applying the right-hand screw rule to moving charges, it is simply necessary to replace the (Il) vector shown in Fig. 3 by the $(Q\mathbf{v})$ vector and think of turning the $(Q\mathbf{v})$ vector to the **B** position; the right-hand travel of this operation defines the direction of the force exerted on the moving charge. This operation is illustrated in Fig. 6 where the magnetic (or **B**) field is indicated as being directed out of the plane of the page by the · symbols. Simply thinking of turning the **v** vector to coincide with

the **B** vectors produces the right-hand rotation which defines the direction of **f**. In the case shown in Fig. 6 one thinks of turning the **v** vector out of the page through an angle of 90° in order that it coincide in direction with the **B** vectors which are directed out of the page.

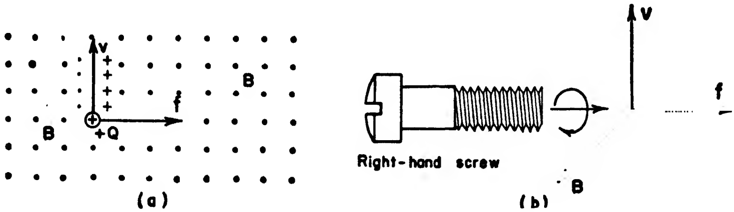


FIG. 6. Think of turning the **v** vector into the **B** vector position (toward the reader) to obtain the right-hand screw direction which gives the direction of the force.

In finding the forces on moving negative charges, it is usually simpler to reverse the direction of **v** (that is, reverse the direction of travel) and then treat the resulting $(-Q)(-\mathbf{v})$ as a positive $(Q\mathbf{v})$ for which the various rules have been formulated than it is to formulate another set of rules. *A negative charge moving in any direction (say, in the* $+\mathbf{v}$ *direction) is equivalent to a positive charge moving in the* $-\mathbf{v}$ *direction.*

If, in the plane in which $(Q\mathbf{v})$ and **B** are located, **B** is not directed at right angles to the direction of the vector $(Q\mathbf{v})$,

$$f = B \sin \theta \Big]_{\mathbf{B}}^{(Qv)} (Qv) \tag{9-a}$$

as illustrated in Fig. 4 if $(Q\mathbf{v})$ is thought of as replacing $(I\mathbf{l})$.

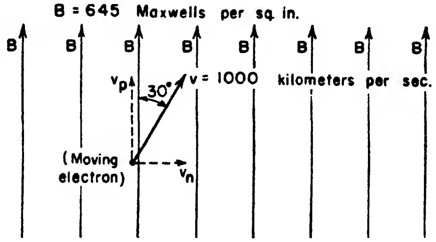


FIG. 7. An electron moving across a magnetic field in the manner shown will spiral about a vertical axis. (See Prob. 20.)

Example 1. *Illustrating the Use of Equation (9-a).* In Fig. 7 is shown an electron which is traveling instantaneously through a magnetic field at an angle of 30° relative to the direction of the **B** vectors. The **v** vector as shown is in the plane of the page, as are the **B** vectors in this case.

The component of **v** which is parallel to the **B** direction produces no magnetic force, and we need consider only the component of **v** which is normal

to the direction of the **B** vectors, namely, v_n in Fig. 7. Since the $\sin\theta$ term in equation (9-a) may be associated with either **B** or **v** to obtain the effective components, we here choose to associate this term with the total v to obtain an effective or normal component of **v** of

$$v_n = 10^6 \sin 30° = 5 \times 10^5 \quad \text{m/sec}$$

Magnetic flux density is often stated in the hybrid units indicated in Fig. 7, namely, maxwells per square inch or "lines" per square inch. Until *maxwells* are considered more properly as a unit of measure of magnetic flux, we shall simply note that (No. of) webers/maxwell $= 10^{-8}$ or 1 weber $\equiv 10^8$ maxwells. Hence we make the required transformation to mks units:

$$\frac{\text{webers}}{\text{sq m}} = \frac{\text{maxwells}}{\text{sq in.}} \times \frac{\text{webers}}{\text{maxwell}} \times \frac{\text{sq in.}}{\text{sq m}}$$

$$= 645 \times 10^{-8} \times \frac{10^4}{6.45} = 10^{-2} \quad \text{webers/sq m}$$

The instantaneous force exerted on the electron shown in Fig. 7 is

$$f = BQv_n = (10^{-2})(1.6 \times 10^{-19})(5 \times 10^5) = 8 \times 10^{-16} \quad \text{newton}$$

Thinking of turning $(-\mathbf{v}_n)$ 90° to coincide with **B** tells us that the force is directed into the plane of the page. (It will be remembered that an electron is a negative charge, the magnitude of which is 1.6×10^{-19} coulomb.)

A little consideration will show that the electron in Fig. 7 will not continue to stay in the plane of the page owing to the force which directs it into the plane of the page. The velocity **v** shown in Fig. 7 acquires a component which is directed into the page, and this component of velocity associated with the **B** vectors causes a spiraling motion, the details of which are left to the reader. (See Prob. 20.)

Example 2. *Electrons in a Cathode-Ray Tube.* Cathode-ray tubes which utilize magnetic deflection are used extensively in television systems and elsewhere. Without going into detail, we shall assume that the construction of the tube is such that electrons, in a narrow beam, travel from the electron emitter shown in Fig. 8 to the viewing screen along an undeflected path unless these moving charges encounter a region where a magnetic field exists.

Actually, the electron path is directed through a magnetic field, the intensity and time variation of which is controlled by current-carrying coils that are placed outside the glass envelope of the tube. It will be left for the reader to check the direction of deflection of the beam of electrons shown in Fig. 8.

Let it be assumed that the electrons enter the magnetic field shown in Fig. 8 with a velocity of 1.8×10^7 m/sec, which is the order of magnitude normally encountered in cathode-ray tubes having 1000-volt power supplies. If electrons having this velocity enter a magnetic field so that $(Q\mathbf{v})$ is normal to the direction of the flux density vectors as shown in Fig. 8, then for a specified

flux density (of, say, 0.01 weber/sq m) the magnetic force developed on each of the electrons is

$$f = BQv = (0.01)(1.6 \times 10^{-19})(1.8 \times 10^{7}) = 2.88 \times 10^{-14} \quad \text{newton}$$

This force deflects the electron somewhat as indicated in Fig. 8.

Although the force is numerically small and acts on the electron for only the period of time in which the electron is in the magnetic field, the overall effect on a mass of 9.1×10^{-31} kg will be significant. The desired effect is to make the fluorescent spot (where the electron beam strikes the viewing screen) describe a luminous path on the viewing screen which is a visual

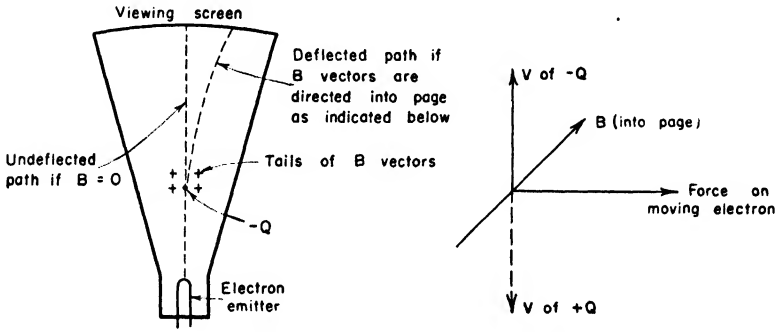


FIG. 8. Illustrating the basic principle of magnetic deflection of a beam of electrons in a cathode-ray tube.

replica of the time variation of the current in the exciting (or mmf) coils which produce the magnetic field. Thus we *see* the time variation of this current by way of the forces developed on moving charges.

4. Motor Action. Practically all electric motors operate on the principle of magnetic forces developed on current-carrying conductors, that is, the *BIl* principle.[1] The magnetic field is usually established by coils of current-carrying conductors wound on iron cores located on the stationary member of the motor. The current-carrying conductors on which the magnetic forces are developed are usually placed on the movable member, and whether or not they are placed on the movable member they are called the *armature conductors*.

The basic principle of motor action (or *BIl* effect) is illustrated in

[1] Small motors like those used in electric clock mechanisms and in some electric shavers often employ pieces of iron in place of the more conventional current-carrying conductors. The pieces of iron (or poles) are, however, reducible to current loops, and even though these equivalent current loops are difficult to evaluate in practice the principle of operation of these small motors is not fundamentally different from that of the larger and more conventional motors.

Fig. 9, where the 40-cm cross bar represents the armature conductor. The armature conductor of this hypothetical motor is supplied with a current from battery E through conducting rails as indicated and is located in a magnetic field, the $\mathbf{B}$ vectors of which are shown to be directed into the plane of the page. The magnitude of $\mathbf{B}$ is specified in "lines" per square inch in order to introduce this commonly used unit of measurement. The term "line" refers to a cgs line or *maxwell* of magnetic flux. The number of webers per maxwell is 10^{-8}, or one weber (as a unit of flux measurement) is 10^8 times as large as a maxwell. Before lines per square inch (or maxwells per square inch) can be used in mks equations they must, of course, be transformed in one way or another to webers per square meter. This transformation may be made in an orderly sequence of steps as shown below, or it may be made in any haphazard

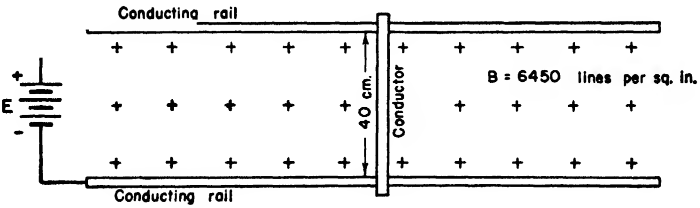


FIG. 9. A linear velocity motor.

manner provided the *number of* lines per square inch specified is transformed to the equivalent *number of* webers per square meter. One orderly method of transformation is

$$\frac{\text{(No. of) webers}}{\text{sq m}} = \frac{\cancel{\text{lines}}}{\cancel{\text{sq in.}}} \times \frac{\text{webers}}{\cancel{\text{line}}} \times \frac{\cancel{\text{sq in.}}}{\text{sq m}}$$

$$= 6450 \times 10^{-8} \times \frac{10^4}{6.45} = 0.10 \quad \text{weber/sq m}$$

If it is assumed that the armature conductor of Fig. 9 is supplied with a current of 60 amp, the force developed on this conductor is

$$f = BIl = (0.10)(60)(0.40) = 2.4 \text{ newtons} \quad \text{(or 0.54 lb)}$$

and the mechanical work done by this force in moving through a distance of D equal to 5 m, say, is

$$W_{\text{mech.}} = fD = 2.4 \times 5 = 12 \quad \text{newton-meters or joules} \quad \text{(8.83 lb-ft)}$$

If we know that, as a result of the type of load which is connected to the armature conductor, the conductor completes the journey of 5 m

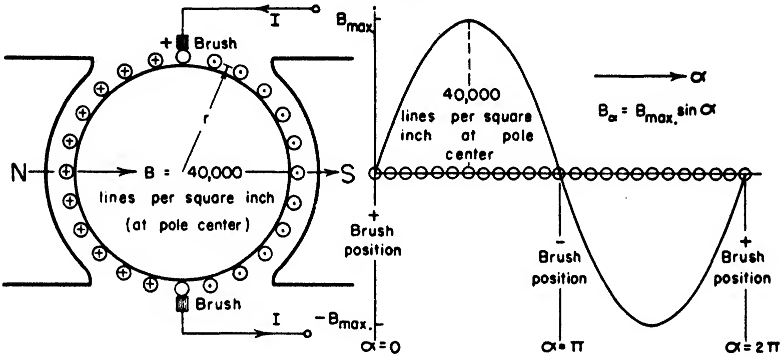
along the rails in a period of 0.8 sec, the time-averaged power developed by this motor is

$$P_{\text{mech.}} = \frac{W_{\text{mech.}}}{t} = \frac{12}{0.8} = 15 \quad \text{joules/sec or watts}$$

Expressed in horsepower, this motor develops 15/746 or about 0.02 hp since

$$\text{(No. of) hp/watt} = \frac{1}{746} \text{ or } 1 \text{ hp} \equiv 746 \text{ watts}$$

It is left for the reader to determine in which direction the motor runs for the battery polarity shown in Fig. 9 and to decide what the effect



(a) CROSS-SECTIONAL VIEW OF AN ELEMENTARY TWO-POLE MOTOR. EACH CONDUCTOR IS l UNITS INTO THE PAGE.

(b) DEVELOPED DIAGRAM OF CURRENT-CARRYING CONDUCTORS LOCATED IN A MAGNETIC FIELD, THE DENSITY OF WHICH IS SINUSOIDALLY DISTRIBUTED

Fig. 10.

would be of always reversing the battery polarity at the end of each journey of D meters of travel. See Prob. 13.

A more practical type of motor than the one described above is pictured in cross-section in Fig. 10-a where the armature conductors indicated by the ⊕ and ⊙ symbols are mounted on the movable or rotating member of the machine. In this case, one-half the total armature current (I in Fig. 10-a) is directed to the conductors facing the north pole and the other half is directed to the conductors facing the south pole, through a sliding contact arrangement called a commutator. The details of this arrangement are not important here except to note that the commutator must obviously furnish the armature conductors facing a north pole with current which is oppositely directed to the current which is furnished to armature conductors facing a south pole if the motor is to run.

A study of Fig. 10-a will show that if the armature conductors facing the north pole are always fed with current as indicated (regardless of rotation), these conductors will all tend to develop counterclockwise rotation, as will the belt of conductors facing the south pole if they are continuously fed with current which is directed out of the page.

Since the space distribution of the magnetic flux density is not uniform from brush to brush, allowance for this fact must be made in determining the time-averaged force which is developed on each conductor as it travels from brush to brush. If uniform speed of rotation is assumed, the time-averaged force on a conductor is determined simply as $B_{\text{av.}}I'l$ since B is the only variable involved; the symbol I' represents one-half the total armature current since each conductor receives only one-half the total armature current, I in Fig. 10-a.

The torque developed by the motor illustrated in Fig. 10-a is

$$T = N(B_{\text{av.}}I'l)r \quad \text{m-newtons} \tag{11}$$

where N is the total number of armature conductors

$B_{\text{av.}}$ is the space-averaged value of B under one pole face expressed in webers per square meter

I' is the current per conductor expressed in amperes

l is the axial length of the conductors expressed in meters

r is the radius at which the $BI'l$ force acts, expressed in meters.

The mechanical energy (or work) developed by this motor, expressed in terms of the circumferential displacement d of any one conductor in time Δt, is

$$\begin{aligned} W_{\text{mech.}} = fd &= f(2\pi r \times \text{rps} \times \Delta t) \\ &= fr[2\pi(\text{rps})]\,\Delta t \\ &= T\omega\,\Delta t \end{aligned} \tag{12}$$

where T is torque, ω is the angular velocity expressed in radians per second, and Δt is the period of time in seconds required for one conductor to travel the circumferential distance d which was employed in the derivation of equation (12).

Since motor power is the time rate of the motor's ability to do work,

$$P_{\text{mech.}} = \frac{W_{\text{mech.}}}{\Delta t} = T\omega \tag{13}$$

where, if torque T is expressed in meter-newtons and angular velocity ω in radians per second, $P_{\text{mech.}}$ is given in watts.

Equation (13) expresses the power of a rotating machine much more concisely than do some of the hybrid expressions often used for this

purpose. One of these hybrid expressions with which the reader is undoubtedly familiar reads

$$P_{\text{mech.}} = \frac{2\pi r(\text{rpm})f}{33{,}000} \quad \text{hp}$$

In electrical calculations even mechanical power is usually handled in watts, and the conversion to horsepower is made later (if the occasion demands it) on the basis of 746 watts being the equivalent of 1 hp.

The manner in which motor performance is calculated in a case where the flux density space distribution is sinusoidal (as pictured in Fig. 10-b) is outlined in the following example. In practice, the space distribution of the flux density is usually much more rectangular in shape than the smooth sine wave distribution shown in Fig. 10-b. The latter is used here since it can be averaged without resorting to point-by-point graphical methods.

Example. *D-C Motor Calculations.* Let it be required to find the torque and power developed by the d-c motor armature shown in Fig. 10 and the efficiency of this armature if it is so connected to a mechanical load that it revolves at 3000 rpm. The known data are

Voltage applied to the brushes, $V = 36$ volts
Total armature current (I in Fig. 10-a) $= 20$ amp
Total number of armature conductors $N = 24$
Axial length of each conductor $l = 20$ cm
Radius at which each conductor is from axis of rotation, $r = 10$ cm
Space distribution of flux density as shown in Fig. 10-b.

The maximum value of B (40,000 lines/sq in.) is first converted to the corresponding number of webers per square meter as previously done in this chapter. Thus

$$B_{\text{max.}} = 40{,}000 \times 10^{-8} \times \frac{10^4}{6.45} = 0.62 \quad \text{weber/sq m}$$

Since $B_\alpha = B_{\text{max.}} \sin \alpha$ as indicated in Fig. 10-b, the average value of B from brush to brush is

$$B_{\text{av.}} = \frac{1}{\pi} \int_0^\pi B_{\text{max.}} \sin \alpha \, d\alpha = \frac{B_{\text{max.}}}{\pi} \Big[-\cos \alpha \Big]_{\alpha=0}^{\alpha=\pi} = \frac{2}{\pi} B_{\text{max.}}$$

In this case

$$B_{\text{av.}} = 0.637 \times 0.62 = 0.395 \quad \text{weber/sq m}$$

From equation (11),

$$\text{torque } T = NB_{\text{av.}} I'lr = (24)(0.395)\left(\frac{20}{2}\right)(0.2)(0.1)$$

or

$$T = 1.895 \text{ m-newtons} \quad \text{(using a slide rule)}$$

The speed of rotation is reduced to radians per second:

$$\omega = \text{rpm} \times \frac{\text{rps}}{\text{rpm}} \times \frac{\text{rad}}{\text{rev}} = 3000 \times \frac{1}{60} \times 2\pi = 100\pi \quad \text{radians/sec}$$

Hence the power developed by the motor is

$$P_{\text{mech.}} = T\omega = 1.895 \times 100\pi = 595 \quad \text{watts} \quad (\text{or } 0.798 \text{ hp})$$

The input to the motor armature is the product of the voltage applied to its terminals and the current which enters and leaves these terminals. The input power in this case is $VI = 36 \times 20 = 720$ watts, and so the armature efficiency η is

$$\eta = \frac{\text{watts output}}{\text{watts input}} = \frac{595}{720} = 0.827$$

The reason that the motor does not convert its entire input (720 watts) to mechanical power is that the current flowing through the resistance of the

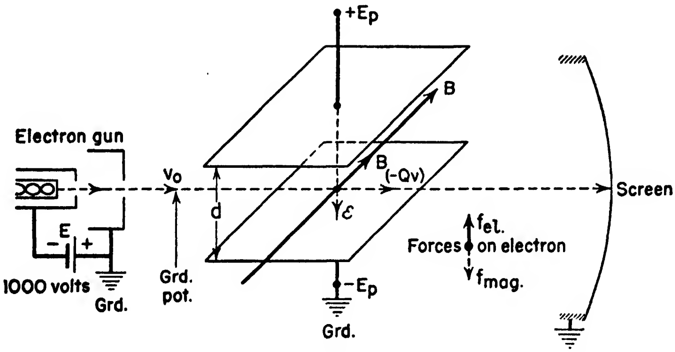


FIG 11. Oppositely directed forces are developed on electron at plate center.

brushes and armature conductors develops heat power losses (Ri^2's) which, of course, do not contribute to the mechanical power developed by the armature. The efficiency of large armatures operating at high voltages may be greater than 0.98.

5. Moving Charges under the Influence of Both ℰ and B Fields. (1) *Determination of the Q_e/m_e Ratio of the Electron.* In Fig. 11 is shown a somewhat more detailed diagram of the cathode-ray tube than that shown in Fig. 8; although several of the refinements whereby a narrow beam of electrons is obtained at the v_0 position are still not shown in Fig. 11. The essential feature of the electrodes shown to the left of the v_0 position is that these electrodes (which are sometimes referred to as the electron gun) produce a narrow beam of electrons at the v_0 position.

As shown, the region in the vicinity of the v_0 position is essentially at ground potential, so that the velocity which the electrons have at this position remains essentially constant from the v_0 position to the screen unless these electrons are acted upon by either magnetic forces or electric forces or both.

The velocity of an electron at the v_0 position shown in Fig. 11 may be determined from a knowledge of the potential difference between this position and the position in the electron gun where the velocity of the electron is essentially zero. This potential difference is indicated in Fig. 11 as E and, for the sake of illustration, a numerical value of 1000 volts has been assigned to this potential difference. It will be observed that the positive electrode of the "gun" is grounded and that the other electrode which is connected to E is held at -1000 volts or 1000 volts *below* ground potential. (The path whereby the electrons return from the screen to the electron emitter is not shown since it tends to complicate the diagram.)

At the $-E$ volt position in the electron gun, an electron has a potential energy relative to ground potential of EQ_e joules, where Q_e is the charge of the electron. In losing this potential energy as it travels from the $-E$ position to the v_0 position (or ground potential region) an electron gains a corresponding amount of kinetic energy. From this fact, the velocity of the electron at the v_0 position may be calculated since

$$0.5m_e v_0^2 = EQ_e \tag{14}$$

or

$$v_0 = \sqrt{\frac{2EQ_e}{m_e}} \tag{15}$$

where m_e is the mass of the electron.

If, instead of permitting the electron to travel to the screen undisturbed at velocity v_0, we apply a potential difference E_p to the parallel plates shown in Fig. 11, the electron will find itself in an electric field as it passes between these plates. The magnitude of the $\mathcal{E}$ vector is readily calculated as E_p/d, where d is the separation of the parallel plates. Hence the electric force developed on the electron is

$$f_{\text{el.}} = \mathcal{E}Q_e = \frac{E_p}{d}Q_e \quad \text{(newtons, in mks units)} \tag{16}$$

This force will be directed in the $-\mathcal{E}$ direction since the electron is a negative charge. If this were the only force acting on the electron from the v_0 position to the screen, the electron would strike the screen at a position somewhat above the point shown in Fig. 11. Assume, however,

that a magnetic field is established by current-carrying coils, the faces of which are normal to the **B** vector shown in the illustration. An electron crossing this magnetic field normally will, of course, experience a magnetic force which is equal to

$$f_{\text{mag.}} = BQ_e v_0 \quad \text{(newtons, in mks units)} \tag{17}$$

In finding the direction of this magnetic force, we form a $+(Q\mathbf{v})$ vector position by reversing the $(-Q\mathbf{v})$ vector shown in Fig. 11, since the moving charge is a negative one in this case. Then, thinking of turning the $+(Q\mathbf{v})$ vector to the **B** direction gives a right-hand screw motion which is directed downward, thus telling us that the direction of the magnetic force is downward or along the $+\mathcal{E}$ direction. Thus $f_{\text{mag.}}$ is directed exactly opposite to $f_{\text{el.}}$.

It is possible to adjust the magnitude of either $\mathcal{E}$ or **B** to make $f_{\text{mag.}} = f_{\text{el.}}$, and when this adjustment is made the luminous spot on the screen is brought back to the position it has when neither of the two forces is acting on the electron beam. Under balanced conditions ($f_{\text{mag.}} = f_{\text{el.}}$) it follows from equations (16) and (17) that

$$BQ_e v_0 = \frac{E_p}{d} Q_e \tag{18}$$

and since v_0 is known from equation (15) to be $\sqrt{2EQ_e/m_e}$ we may write

$$B^2 \frac{2EQ_e}{m_e} = \frac{E_p{}^2}{d^2}$$

or

$$\frac{Q_e}{m_e} = \frac{E_p{}^2}{2Ed^2B^2} \tag{19}$$

Thus is shown one method of measuring the charge-to-mass ratio of the electron in terms of two potential differences, a length d and a magnetic flux density. Assuming that all the quantities on the right-hand side of equation (19) can be measured to two- or three-significant-figure accuracy, the result will be 1.76×10^{11} for the charge-to-mass ratio in mks units. This result corresponds to the ratio Q_e/m_e, where $Q_e = 1.6 \times 10^{-19}$ coulomb and $m_e = 9.1 \times 10^{-31}$ kg, the values which have been used for Q_e and m_e throughout this text.

The measurement of the Q_e/m_e ratio has served to illustrate a situation where both electric and magnetic forces act simultaneously on a moving charge. We are concerned here neither with the details nor with the accuracy of this method of measuring Q_e/m_e as compared with other methods of making the same measurement.

Example 1. Let it be required to find the numerical value of v_0 in Fig. 11 if the value of Q_e/m_e is known to be 1.76×10^{11} and E (in Fig. 11) is 1000 volts.

From equation (15),

$$v_0 = \sqrt{\frac{2EQ_e}{m_e}} = \sqrt{2(1000)(1.76 \times 10^{11})} = 1.88 \times 10^7 \quad \text{m/sec}$$

Example 2. Let it be required to find the magnetic force developed on an electron at the **B** position in Fig. 11 if $B = 2.66 \times 10^{-3}$ weber/sq m, $E = 1000$ volts, and it is assumed that we know $Q_e = 1.6 \times 10^{-19}$ coulomb.

From example (1), $v_0 = 1.88 \times 10^7$ m/sec. for $E = 1000$ volts. Hence

$$f_{\text{mag.}} = BQ_ev_0 = (2.66 \times 10^{-3})(1.6 \times 10^{-19})(1.88 \times 10^7) = 8 \times 10^{-15} \text{ newton}$$

Example 3. Let it be required to find the electric force developed on an electron at the $\mathcal{E}$ position in Fig. 11 if $E_p = 500$ volts and $d = 1$ cm.

$$f_{\text{el.}} = \mathcal{E}Q_e = \frac{E_p}{d}Q_e = \frac{500}{0.01} \times 1.6 \times 10^{-19} = 8 \times 10^{-15} \quad \text{newton}$$

The arrangement shown in Fig. 11, if operated as specified in these examples, would give a null indication of the luminous spot.

(2) *Description of the Cyclotron.* The cyclotron is a massive electromagnetic device in which positively charged particles (usually hydrogen nuclei) are accelerated, step by step, to extremely high velocities. After the particles have acquired millions of electron volts of kinetic energy[2] they are directed into the atomic nuclei of chemical elements with a view toward effecting transmutations of these elements. (An example of one type of chemical transmutation which can be effected in this manner is shown in Fig. 13, page 47.)

The basic operating principle of the cyclotron is diagrammed in Fig. 12 where in (*a*) only one of the two hollow electrodes (called *D* plates) are shown. The relative positions of the two hollow electrodes to one another may be visualized by thinking of sawing a flat tin can along a diameter to form the two electrodes. The region in which the electrodes

[2] A freely moving particle which has a charge equal to that of the magnitude of an electronic charge (1.6×10^{-19} coulomb) acquires 1 electron volt of energy (QE) if it falls through a potential drop of 1 volt. An electron ($-Q_e$) in Fig. 11, for example, falls through a potential difference of -1000 volts in going from the $-E$ electrode to the v_0 or $+E$ position in the cathode-ray tube. As a result of its loss in potential energy the electron gains $(-Q_e)$ (-1000) or $+1000$ electron volts of kinetic energy because the definition employed in specifying electron volts of energy reckons Q_e as unity. The 1000 electron volts of kinetic energy gained by the electron in Fig. 11 is very small compared to the tens or hundreds of millions of electron volts of kinetic energy developed by the cyclotron and other particle accelerators employed in nuclear physics.

are located is pierced by a magnetic field as indicated, and the two electrodes are energized electrically with a potential difference which reverses its polarity in synchronism with the period of time required

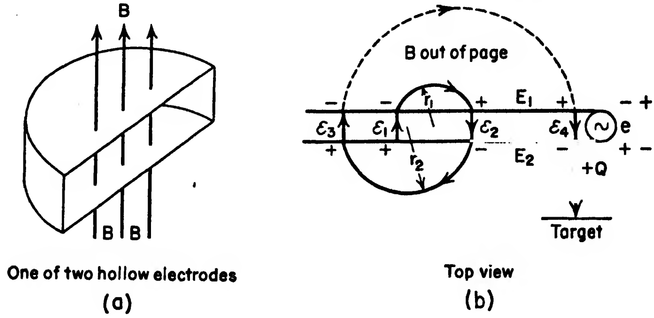


FIG. 12. Illustrating one electrode and a top view of the spiraling path which is executed by $+Q$ in the interior of a cyclotron.

for the charged particle to complete a semicircular journey within the interior of one of the electrodes.

The details of the high-voltage, high-frequency oscillator employed for the purpose of alternately reversing the polarity of the $\mathcal{E}$ field are not important here. The reader may, if he chooses, think simply of the e generator shown in Fig. 12-b as a battery, the polarity of which can be alternately reversed to synchronize with the period of time required for a charged particle to execute one-half cycle of its spiral journey.

To start a cycle of events which takes place within the cyclotron, we may imagine that the positively charged particles are injected into the gap between the two hollow electrodes at the $\mathcal{E}_1$ position at the instant that this electric field vector is directed as shown in Fig. 12-b. $\mathcal{E}_1$ forces these positively charged particles in the direction indicated on the $\mathcal{E}_1$ vector, that is, from $+$ to $-$. Some of these charged particles may be collected by the $-E_1$ electrode, but the particles which are midway between the top and bottom surfaces of the hollow electrode are essentially freely moving particles once they are between the flat surfaces maintained at the same potential. These freely moving particles are, of course, subjected to magnetic (or BQv) forces which cause them to execute a semicircular path within the interior of the E_1 electrode. That this path is semicircular may be seen from an inspection of Fig. 13 if it is recognized that, since v is constant

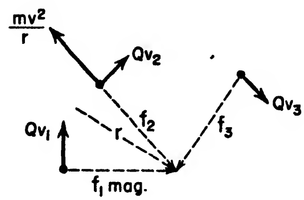


FIG. 13. Illustrating that, for $mv^2/r = BQv$, the path executed is circular where $v_1 = v_2 = v_3$.

within the interior of the electrode, the equilibrium condition

$$\frac{mv^2}{r} = BQv \tag{20}$$

demands that the radius of gyration r be constant. (It will be remembered that the centrifugal or outwardly directed force on a mass m which is not traveling in a straight line is mv^2/r where r is the radius of curvature of the path. It must also be remembered that the BQv force in this case does not increase v since it is normal to $\mathbf{v}$ at all places.) The radius of the semicircular path taken by the particles after they leave the accelerating force due to $\mathcal{E}_1$ is, from equation (20),

$$r_1 = \frac{mv_1}{BQ} \tag{21}$$

After completing the approximate semicircular journey within the interior of the E_1 electrode, the particle enters the $\mathcal{E}_2$ gap where $\mathcal{E}_2$, as a result of reversal of the generator polarity, is so directed as to increase the velocity to v_2 which is greater than v_1, the velocity with which the particle enters the $\mathcal{E}_2$ gap. This increase in velocity is the result of the particle (already at v_1 velocity) falling through a second potential drop which is nicely timed to be present in the $\mathcal{E}_2$ gap with the polarity as shown at the instant the particle arrives at the $\mathcal{E}_2$ gap. Hence

$$r_2 = \frac{mv_2}{BQ} \tag{22}$$

is greater than r_1 because of the increased velocity.

The particle spirals outwardly as shown in Fig. 12-b with ever-increasing velocities (and hence kinetic energies) until it has sufficient energy to shatter some atomic nuclei and effect certain chemical transmutations.

The fact that it takes the particle precisely the same period of time to complete any one of its semicircular excursions (regardless of the ever-increasing radius) makes it possible to employ an alternating potential difference of fixed frequency to effect the necessary reversals of the $\mathcal{E}$ field. The time required to execute the first semicircular portion of the path shown in Fig. 12-b is

$$T_1 = \frac{\pi r_1}{v_1} \quad \text{(since } v_1 \text{ is constant)} \tag{23}$$

and the time required to execute the second semicircular portion of the path is

$$T_2 = \frac{\pi r_2}{v_2} \quad \text{(since } v_2 \text{ is constant)} \tag{24}$$

For T_1 to equal T_2 it is necessary that the r/v ratios be equal, and from equations (21) and (22) it may be seen that

$$\frac{r_1}{v_1} = \frac{r_2}{v_2} = \frac{m}{BQ} \tag{25}$$

If we neglect any change in the m/Q ratio which might arise as a result of the particle attaining velocities which are comparable to the velocity of light (3×10^8 m/sec), the right-hand member of equation (25) is constant and T_1 and T_2 of equations (23) and (24) are equal.[3]

Example. Consider a cyclotron in which the hydrogen nuclei are given increases in kinetic energy of 36,000 electron volts each time these particles cross the gap between the two hollow electrodes. (This means simply that the nuclei which have the same charge as the electron, except for sign, fall through a potential difference of 36,000 volts each time they cross the gap; the potential difference being supplied to the gap by the a-c generator or oscillator shown in Fig. 12-b.)

Let it be required to determine the kinetic energy of one of the nuclei and the radius of the path, r_2, as the particle *enters* the $\mathcal{E}_3$ gap in Fig. 12-*b* if the particle enters the $\mathcal{E}_1$ gap at essentially zero velocity and the magnetic flux density is constant at 0.80 weber/sq m. It is known that to two significant figures the mass of the hydrogen nucleus is 1.7×10^{-27} kg, and the charge is 1.6×10^{-19} coulomb.

During its passage across the $\mathcal{E}_1$ gap the nucleus acquires a velocity of

$$v_1 = \sqrt{\frac{2EQ}{m}} = \sqrt{\frac{2(36{,}000)(1.6 \times 10^{-19})}{1.7 \times 10^{-27}}} = 2.60 \times 10^6 \text{ m/sec}$$

in accordance with equation (15). This is the velocity which is associated with a particle having a mass of 1.7×10^{-27} kg and a kinetic energy of 36,000 electron volts.

The radius of the first semicircular portion of the path is

$$r_1 = \frac{mv_1}{BQ} = \frac{(1.7 \times 10^{-27})(2.60 \times 10^6)}{(0.8)(1.6 \times 10^{-19})} = 3.46 \times 10^{-2} \text{ m} \quad \text{(or 3.46 cm)}$$

in accordance with equation (21). Any particle having the same mv/Q which entered a magnetic field having a density of 0.8 weber/sq m at right angles to the **B** direction would likewise be forced to follow a circular path of 3.46-cm radius.

Upon reaching the $\mathcal{E}_2$ gap, the nucleus which is being investigated acquires an additional 36,000 electron volts of energy, and now it has a total of 72,000

[3] The relativistic changes in mass are small for velocities less than 5×10^7 m/sec, and in any case have little place in a beginning text. Even neglecting this change of mass, a hydrogen nucleus which finally acquires 10 million electron volts of kinetic energy has a velocity of only about 4.3×10^7 m/sec as determined from $0.5mv^2 = EQ$.

electron volts of kinetic energy. The velocity which is associated with 72,000 electron volts of kinetic energy is

$$v_2 = \sqrt{\frac{2EQ}{m}} = 1.41 \times v_1 = 3.68 \times 10^6 \text{ m/sec}$$

and the corresponding radius of the v_2 path is

$$r_2 = 1.41 r_1 = 4.88 \times 10^{-2} \text{ m} \quad (4.88 \text{ cm})$$

Thus it is that the positively charged hydrogen nucleus (which is initially obtained by stripping a normal hydrogen atom of its one orbital electron) is spiraled outwardly in the cyclotron and finally directed on to the target as shown in Fig. 12-b.

6. Summary. The subject of magnetically developed forces may be summarized briefly as: **Current-carrying conductors and moving charges in the presence of a magnetic field experience (BIl) and (BQv) forces respectively.**

The magnetic vector **H** is useful in force calculations only in so far as it provides an analytical means of evaluating the *magnetic force* vector **B**. ($\mathbf{B} = \mu_0\mu_r\mathbf{H}$).

Magnetic flux density **B** may be considered as the force per unit ($I\mathbf{l}$) or the force per unit ($Q\mathbf{v}$) with due regard for the angular displacements between the directions of the vectors **B** and ($I\mathbf{l}$) and between **B** and ($Q\mathbf{v}$).

An additional unit of measurement of magnetic flux (the maxwell or the cgs line of flux) has been introduced because it is so widely used in both engineering and physics and because we shall soon have occasion to use this unit of magnetic flux in connection with ferromagnetic circuits where the published data (in the form of curves) is almost universally presented in terms of maxwells (or lines) per square centimeter or in terms of maxwells (or lines) per square inch.

Only two new relationships (which are viewed as fundamental laws of nature) have been introduced in this chapter:

(1) Magnetically developed force,

$$f_{\text{mag.}} = B_n I l = B I l_n$$

where B_n is the component of **B** which is normal to **l** or
l_n is the component of **l** which is normal to **B**.

(2) Magnetically developed force,

$$f_{\text{mag.}} = B_n Q v = B Q v_n$$

where B_n is the component of **B** which is normal to **v** or
v_n is the component of **v** which is normal to **B**.

The direction in which $\mathbf{f}_{\text{mag.}}$ acts is obtained by observing the direction of right-hand screw travel which results from turning the vector $(I\mathbf{l})$ or $(Q\mathbf{v})$ through the smaller angle to the **B** position in space.

In the last section of this chapter and in a few of the problems which follow, some use is made of the fact that the centrifugal force acting on a physical body which is moving along a curved path is

$$f_{\text{centrifugal}} = \frac{mv^2}{r}$$

where m is the mass, v the velocity, and r the distance between mass m and the center of gyration. It is presumed that the reader has encountered the derivation of this relationship in his study of physics or mechanics.

PROBLEMS

1. What is the magnitude of the magnetic force developed on a 50-cm section of the current-carrying conductor shown in Fig. 1, page 229, if the magnitude of the current is 150 amp and the magnitude of the flux density is 0.08 weber/sq m? Are the force vectors shown in Fig. 1 properly directed for the specified directions of **B** and $I\mathbf{l}$?

2. It is known from experimental measurements that a magnetic field (which is directed northward and downward at an angle of 45° from the ceiling of a room) is uniformly distributed throughout the region occupied by the room and that the magnitude of the **B** vector is 0.05 weber/sq m. A single-turn rectangular coil (5 m x 1 m) is placed in a vertical east-west plane with the short sides perpendicular to the floor and this coil carries a current of 100 amp.

(*a*) What is the magnitude of the force experienced by each of the long sides?

(*b*) What is the magnitude of the force experienced by each of the short sides?

3. Refer to the current-carrying coil described in Prob. 2. Imagine yourself at the west end of the room in the plane of the rectangular coil. As you view the upper coil side in cross-section you see current directed toward you.

(*a*) Draw a cross-sectional view of the upper coil side as you view the situation, indicating the current direction by means of an appropriate symbol, the **B** direction, and the **f** direction.

(*b*) Draw an end view of the short coil side nearest you and show the directions of $(I\mathbf{l})$, **B**, $\mathbf{B}_n$, and indicate by symbol or state the direction of the force on the short coil side relative to your position.

4. (*a*) Check the direction of the repelling magnetic force shown in Fig. 5-b, page 231, by (1) the cross-sectional view method shown in Fig. 2, page 229; (2) the right-hand screw rule.

(*b*) Determine the magnitude of this repelling force which is developed over a 50-cm section of conductor near a point midway between the supports if $I = 500$ amp and the distance of separation of the two currents in this section (center-to-

center) is 2 cm. The assumption may be made that the 50-cm section is a very small fraction of the total length of the conductors and that over the 50-cm section the wires are straight and parallel.

5. A 100-m length of straight wire carrying 150 amp is stretched horizontally above the surface of the earth with the axial length directed from east to west, the direction in which the current flows. The earth's magnetic field in this locality is directed normally to the (Il) direction and has a horizontal density component of 2×10^{-5} weber/sq m (directed northward) and a vertical component of 4×10^{-5} weber/sq m (directed downward).

(*a*) Find the horizontal component of the magnetic force developed on the current-carrying wire and specify its direction, north or south.

(*b*) Find the vertical component of the force and specify its direction, up or down.

(*c*) What is the magnitude of the resultant force developed on the wire?

6. Refer to Fig. 9, page 236. Find the magnitude and direction of the force developed on the 40-cm conductor if battery E supplies this conductor with 120 amp. (The + symbols indicate that the flux density vectors are uniformly distributed and directed into the plane of the page, and it will be remembered that "lines" refer to maxwells of magnetic flux.)

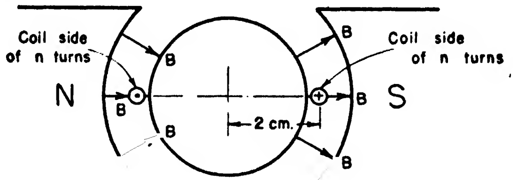


FIG. 14. Driving element of a permanent-magnet type of instrument. (See Prob. 9.)

7. Consider the case where an electron is in stable equilibrium $[(mv^2/r) = BQv]$ in a circular path with the **v** vector always directed normally to the direction of the **B** vector. If the magnitude of **v** is 10^8 cm/sec and the magnitude of **B** is 0.0562 weber/sq m, find:

(*a*) The radius of the circular path.

(*b*) The angular velocity expressed in radians per second.

(*c*) The revolutions per second.

8. A hydrogen nucleus ($Q = 1.6 \times 10^{-19}$ coulomb and $m = 1.7 \times 10^{-27}$ kg) is in stable equilibrium $[(mv^2/r) = BQv]$ when it is spinning in a circle which is 1 ft in radius in a magnetic field, the flux density of which is 0.5 weber/sq m.

At what number of revolutions per second is the charged particle moving about the circular orbit?

9. A cross-sectional view of the driving element of a permanent magnet (or D'Arsonval) type of instrument is shown in Fig. 14. The effective axial length of the conductors (only two of which are illustrated) is 4 cm directed into the plane of the page. The magnetic flux density in the air gap is uniformly distributed, directed as shown, and has a magnitude of 0.050 weber/sq m (or 500 maxwells/sq cm).

Find the number of turns of wire (that is, the number of conductors in one coil side) which is required to produce a driving torque of 200 cm-dynes if I = 10 ma.

Specify direction of rotation. *Note:* A dyne is unit of force which is 10^{-5} times smaller as a unit of measure than the newton.

10. Refer to Fig. 15. Assume that the magnetic flux density B is uniformly distributed as shown and has a magnitude of 0.0667 weber/sq m. When the b terminal of the coil is connected to the positive terminal of a supply battery (and the a terminal to the negative terminal) a current of 15 amp flows through the coil.

(*a*) Find the total force exerted on the coil sides which are 30 cm in length when the coil is in the position shown, and specify the direction of rotation (as observed from the terminal end of the coil) which this force would tend to develop if the coil had an axis of rotation along the center dashed line.

(*b*) What is the magnitude of the force developed on the 20-cm ends of the coil in the position shown?

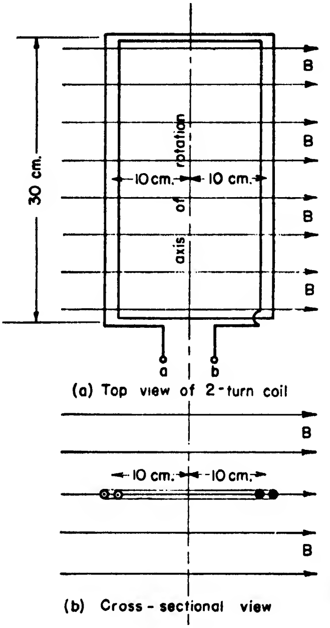


FIG. 15. See Probs. 10 and 11.

11. Determine the torque which is developed by the current-carrying coil described in Prob. 10, Fig. 15, that tends to produce rotation about the central axis, and express the result in meter-newtons and in centimeter-dynes.

If the coil were free to rotate about the central axis, would it revolve continuously under the influence of the magnetically developed torque? Explain. (It is assumed that the coil sides always carry currents directed as shown in Fig. 15-b regardless of the coil position.)

12. (*a*) If it is known that $B = 1290$ maxwells/sq cm, what is the value of B expressed in maxwells per square inch?

(*b*) What is the value of B referred to above expressed in webers per square meter?

13. Refer to the "linear-velocity" motor shown in Fig. 9, page 236. Assume that the armature conductor (the sliding cross bar) is supplied with 120 amp from the battery through a reversing switch which is not shown in Fig. 9. This reversing switch reverses the polarity of the armature current once each second at the instant the armature conductor has reached the end of its 10-m excursion back and forth along the rails. (It is assumed that these 10-m excursions are gradually stopped with spring bumpers or the like which occasion no loss of energy to the working cycle.)

(*a*) How many pound-feet of work might be performed by this motor in the course of a 10-hr day?

(*b*) What horsepower is developed by this motor?

14. Refer to the d-c motor which is shown in cross-section in Fig. 10-a, page 237. If, *instead of* the sinusoidal space distribution of flux density shown in Fig. 10-b, the distribution is as shown in Fig. 16, find the armature torque, the power developed

mechanically, and the efficiency of the armature under the following known conditions:

voltage supplied to the brushes = 110 volts
total current supplied to the armature (I in Fig. 10-a) = 30 amp
total number of armature conductors = 100 (not the 24 shown)
axial length of each conductor = 20 cm
r (in Fig. 10-a) = 10 cm
speed of rotation = 1500 rpm

15. Show qualitatively by means of a diagram that the magnetic forces developed on each of two electrons which are traveling side by side along straight parallel paths (as, for example, in a beam of electrons) tend to nullify the electric forces of repulsion which these charges experience.

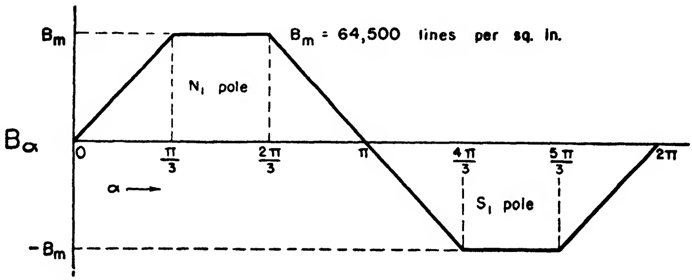


FIG. 16. Graph of B_α versus α for use in connection with Prob. 14.

16. Consider a freely moving electron ($Q_e = -1.6 \times 10^{-19}$ coulomb and $m_e = 9.1 \times 10^{-31}$ kg) which has a velocity of 10^9 cm/sec directed due north in a horizontal plane.

At a point 100 cm from a reference point (x_0, y_0) in the horizontal plane, the electron traveling in the $+y$ or north direction encounters a "wall" of magnetic flux that is directed vertically downward, the magnitude of the flux density of which is 10 maxwells/sq cm.

(*a*) What side thrust is experienced by the electron when it encounters the "wall" of magnetic flux?

(*b*) At what point in the x-y plane does the electron leave the magnetic field?

(*c*) What length of time is required for the electron to travel from the reference point (x_0, y_0) back to the point where the path of the electron crosses an east-west line running through the reference point?

17. Refer to Fig. 17. What is the kinetic energy of the hydrogen nucleus ($Q = 1.6 \times 10^{-19}$ coulomb and $m = 1.7 \times 10^{-27}$ kg) if it possesses a velocity of 5×10^6 m/sec? Express the result in joules and in electron volts of energy.

18. What number of volts of potential drop would be required to give the hydrogen nucleus shown in Fig. 17 its velocity of 5×10^6 m/sec?

19. Assume that the hydrogen nucleus shown in Fig. 17 is moving in a free-space region between the two "walls" of magnetic flux and that no forces act upon this freely moving particle when it is between the two "walls."

(*a*) What will be the nature of the path traversed by H^+?

(*b*) What period of time is required for H^+ again to be midway between the two "walls" after it leaves the position shown in Fig. 17, assuming that $v_0 = 5 \times 10^6$ m/sec?

20. (*a*) Determine the radius of the circular motion produced by v_n in Fig. 7, page 233.

Note: v_n produces a circular motion of the electron about a vertical axis in Fig. 7 which is not influenced by the fact that v_p combined with the circular motion causes a helical path to be traced out by the electron.

(*b*) Find the length of time required for the electron to execute one circular path about a vertical axis in Fig. 7.

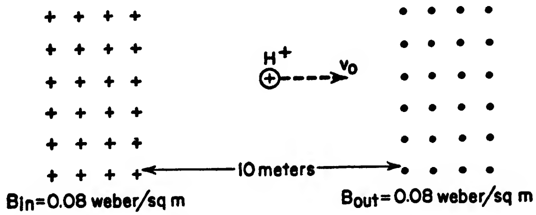


FIG. 17. A hydrogen nucleus midway between two "walls" of magnetic flux. $v_0 = 5 \times 10^6$ m/sec. (See Prob. 17, 18, and 19.)

21. (*a*) How many circular paths are traced out by the spiraling electron in Prob. 20 (Fig. 7) in the length of time required for the electron to travel 1 m in the **B** direction. (Motion in the **B** direction is due solely to v_p.)

(*b*) Show that the pitch of the helix, that is, the unit lengths traveled along the axis of the helix during the time required for the electron to execute one revolution of circular motion, is in general

$$\frac{2\pi}{\omega} v_p$$

where ω is the angular velocity in radians per second produced by v_n.

22. Refer to Fig. 11, page 240. Assume that the Q_e/m_e ratio is known to be 1.76×10^{11} (in mks units), $E = 1000$ volts, $d = 1$ cm, and that B is an unknown quantity.

Experimentally E_p is adjusted to a magnitude of 141.4 volts in order to obtain a null indication of the luminous spot on the viewing screen.

Determine the magnitude of B, neglecting all fringing effects and assuming that **B** is uniformly distributed over the same axial length of the electron beam as is $\mathcal{E}$.

23. Refer to Fig. 12-b, page 244. What is the correct frequency in cycles per second of the e generator (or oscillator) if $B = 0.8$ weber/sq m and hydrogen nuclei are used as the projectiles?

Note: One cycle of frequency includes two reversals of polarity.

24. Refer to Fig. 12-b, page 244. A hydrogen nucleus ($Q = 1.6 \times 10^{-19}$ coulomb and $m = 1.7 \times 10^{-27}$ kg) starts with essentially zero velocity at the + side of the $\mathcal{E}_1$ gap and receives from the electric field 36,000 electron volts of kinetic energy each time it crosses the gap between the two hollow electrodes. If $B = 0.8$ weber/sq m, what length of time is required for the charged particle to appear at the $\mathcal{E}_4$ gap?

25. Refer to the cathode-ray tube diagram in Fig. 11. Assume that the deflecting plates are 1.88 cm long in the $\mathbf{v}_0$ direction and that the distance from the midpoint between the plates (where the $\mathcal{E}$ vector is shown) to the screen is 20.94 cm. The plate-to-plate separation is $d = 1$ cm.

If $E = 1000$ volts as shown, $E_p = 100$ volts, and $B = 0$, at what point will the electron beam strike the screen relative to the undeflected striking point shown in Fig. 11. Neglect fringing and consider the screen as an essentially straight or plane surface.

26. Consider two equal and like charges to be traveling side by side in straight parallel paths which are separated from one another by a distance r. As a fairly reasonable value of the magnetic field intensity due to either of these single moving charges, *assume*

$$H = \frac{Qv}{4\pi r^2} \quad \text{(in rationalized units)}$$

even though this expression cannot be derived readily from any of the concepts given in this text. A free-space medium is postulated.

At what velocity is the electric force of repulsion (based on $D = Q/4\pi r^2$) completely nullified by the magnetic force of attraction?

27. (*a*) What is the magnitude and direction of the magnetic force developed on the 100-m length of wire described in Prob. 5 if the wire is changed in position by rotating it 90° in the horizontal plane in a counterclockwise direction (when viewed from below)?

(*b*) What is the angle between the resultant $\mathbf{B}$ vector and the Il direction in the new position?

28. Show that the magnetic force on a current-carrying conductor may be expressed as $\mathbf{l} \times \mathbf{B}$ per unit current. (See page 16.)

29. Show that the magnetic force on a moving charge may be expressed as $\mathbf{v} \times \mathbf{B}$ per unit charge. (See page 16.)

CHAPTER X

Magnetically Generated Voltages

1. Elementary Dimensional Investigation of Generated Voltages. An elementary dimensional analysis is often helpful in correlating a new concept with concepts which are older or better established in our minds. Electric potential or voltage, for example, has previously been considered as work per unit charge and is therefore expressed dimensionally as

$$E = W^1Q^{-1} = f^1l^1Q^{-1} \tag{1}$$

where W represents work, the product of force times length, and the superscripts attached to all terms on the right-hand sides (including even the first power terms) indicate that the dimensional properties of the left-hand member is under investigation. With proper choice of units these dimensional expressions can often be used directly as working algebraic equations as, for example, equation (1) has been used many times in the form $W = EQ$ joules of work or energy.

A large portion of Chapter IX has been devoted to the development of the concept of

$$f = B^1Q^1v^1 \tag{2}$$

Combining the dimensional expressions given in (1) and (2) yields

$$E = B^1v^1l^1 \tag{3}$$

or, since $v = l^1t^{-1}$,

$$E = B^1l^2t^{-1} = \phi^1t^{-1} \tag{4}$$

where ϕ symbolizes magnetic flux. Equations (3) and (4) are often referred to as the Blv and the $d\phi/dt$ concepts respectively of magnetically generated voltages.

Algebraic working equations which agree with experimental measurements[1] may be formed from these dimensional expressions. The funda-

[1] A straight antenna which is cut by the magnetic flux of a radio wave that is traveling at the velocity of light relative to the antenna is an interesting example. In this case the voltage generated in the antenna per unit length ($\Delta E/\Delta l$) may be calculated in terms of Bv, and the calculated values agree with those obtained experimentally and with those calculated in terms of the electric field intensity $\mathcal{E}$.

mental working equations for voltage gradient and for total voltage, generated by a time-varying magnetic flux are

$$\frac{\Delta E}{\Delta l} = Bv \quad (\text{or } e = Bv\,\Delta l \quad \text{for a single conductor}) \tag{5}$$

$$E = \frac{\Delta \phi}{\Delta t} \quad (\text{or } e = \frac{d\phi}{dt} \quad \text{for a single loop}) \tag{6}$$

where Δl is the incremental length of the single conductor and $\Delta\phi/\Delta t$ is the time-averaged rate at which magnetic flux pierces the faces of a single-turn coil. The lower case letter e indicates that *instantaneous* voltages may be calculated in terms of *instantaneous* velocities or *instantaneous* time rates of change of magnetic flux. Since equations (5) and (6) are fundamental laws of nature, they may be used in any systematic set of units.

It is to be expected that careful consideration will have to be given to the manner in which the angular displacements of the vectors **B** and

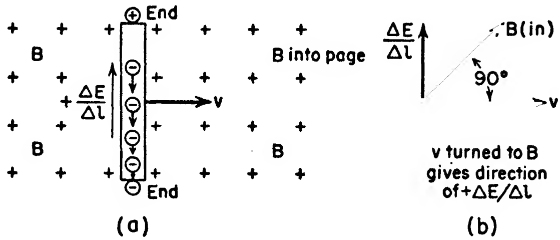


FIG. 1. A conductor of Δl length moving through a magnetic field with velocity v. ($\mathbf{v}$ and $\mathbf{B}$ are at right angles in this case.)

$\mathbf{v}$ affect the magnitude and direction of the vector $\Delta E/\Delta \mathbf{l} = -\mathcal{E}$. Careful consideration will also have to be given to the precise meaning of $\Delta\phi/\Delta t$ (and of $d\phi/dt$) through a closed loop before these expressions can be employed effectively in the solution of problems.

2. The Single-Conductor Concept. It has been shown that electric charges moving relative to a magnetic field experience side thrusts or BQv forces. If, then, a conductor with its many loosely held orbital electrons is moved relative to a magnetic field as shown in Fig. 1, it may be anticipated that these loosely bound negative charges will experience the same magnetically developed forces as if they were individual charges moving relative to the magnetic field. In the case shown in Fig. 1, the negative charges of the metallic conductor are forced to the lower end of the conductor while the positive charges within the molecular structure of the metal are tightly bound to this structure. The result of the motion

of the conductor relative to the magnetic field is that of making the lower end negatively charged which, of course, renders the upper end positively charged since the conductor overall is electrically neutral.

Since the $\mathcal{E}$ vector of electric circuit theory is directed in the + to − direction and since potential rise E is from − to +, it is evident that in Fig. 1

$$+\frac{\Delta E}{\Delta l} = -\mathcal{E} \quad \text{(within the conductor)} \tag{7}$$

Over any particular length of conductor (Δl) which is in a region of uniform flux density, this potential gradient ($\Delta E/\Delta l$) may be evaluated as

$$\frac{\Delta E}{\Delta l} = B \sin \theta\Big]_{\mathbf{v}}^{\mathbf{B}} v = B_n v = B v_n \tag{8}$$

where $\Delta E/\Delta l$ is given in volts per meter if B is expressed in webers per square meter and v in meters per second.

The symbol B_n implies that if the $B_n v$ form of equation (8) is employed, only the component of $\mathbf{B}$ which is normal to the direction of $\mathbf{v}$ is to be used; whereas if we choose to associate the $\sin \theta\Big]_{\mathbf{v}}^{\mathbf{B}}$ term with v, only the component of $\mathbf{v}$ which is normal to the direction of $\mathbf{B}$ is to be used. (The corresponding situation has been considered in some detail in connection with magnetically developed forces.) If the conductor shown in Fig. 1, for example, were to be moved in a plane which is normal to the plane of the page, $\mathbf{v}$ and $\mathbf{B}$ would have the same (or directly opposite) directions, $\sin \theta\Big]_{\mathbf{v}}^{\mathbf{B}}$ would be equal to zero, and no voltage gradient would be developed along the axial length of the conductor.

There are several rules (or artifices) which will help us remember the correct direction of $+\Delta E/\Delta l$ along a conductor without resorting to the details illustrated in Fig. 1-a. For example, the direction of $+\Delta E/\Delta l$ may be found by applying the right-hand screw rule which is illustrated in Fig. 1-b. To find the direction of $+\Delta E/\Delta l$, we think of turning the $\mathbf{v}$ vector through the smaller angle to the $\mathbf{B}$ vector position, and the resulting direction of right-hand screw travel tells us the direction of $+\Delta E/\Delta l$. (It will be remembered from Chapter IX that we turned $\mathbf{v}$ into the $\mathbf{B}$ position to determine the force on a moving charge.) In cases where $\mathbf{B}$ moves and the conductor is stationary relative to the observer, the observer thinks of $\mathbf{B}$ as stationary and the conductor as having the velocity since the rule is so framed.

The voltage rise per unit length or potential gradient is *always normal* to the plane in which $\mathbf{B}$ and $\mathbf{v}$ are located, whether $\mathbf{B}$ and $\mathbf{v}$ are displaced from one another by 90° as shown in Fig. 1-a or by some other angle.

Another scheme which is often employed to tell the correct direction of $+\Delta E/\Delta l$ along a conductor is illustrated in Fig. 2. Here we think of the moving conductor "bending" the flux lines. Then, if the fingers of the right hand are placed around the conductor to coincide in direction with the direction of the "bend," the thumb points in the $+\Delta E/\Delta l$ direction along the conductor. The scheme works nicely on cross-sectional views but otherwise is usually more cumbersome than the right-hand screw rule.

Three situations arise in practice where the single-conductor concept of generated voltages possesses an advantage over the loop concept: (1) In d-c generator windings (where all conductors are connected to all other conductors), it is usually simpler to visualize the individual

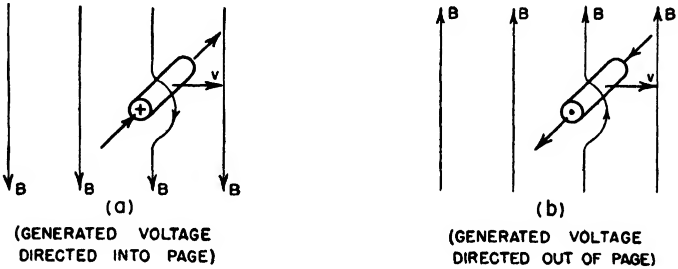


FIG. 2. A scheme for finding the direction of the potential gradient $\Delta E/\Delta l$.

conductors than it is to visualize the two conductors (which are under opposite poles) that form the closed loop. (2) In a straight antenna the magnetic flux cuts across a single conductor, thereby producing *relative* motion between the conductor and the magnetic flux density vector, even though in this case **B** moves and the conductor is stationary. (3) In metal disks revolving in asymmetrically located magnetic fields.

Example 1. The conductor shown in Fig. 1 will be assumed to be moving in the plane of the page at a velocity of 10 m/sec. If $B = 0.05$ weber/sq m and the length of the conductor (Δl) is 3 m,

$$\frac{\Delta E}{\Delta l} = Bv = (0.05)(10) = 0.5 \text{ volt/m} \quad \text{(directed as shown)}$$

and

$$E = 0.5 \times 3 = 1.5 \text{ volts} \quad \text{(potential difference between ends of } \Delta l\text{)}$$

Example 2. Consider a 5-m length of wire mounted in a horizontal plane above the wing of an airplane which is traveling due north at a velocity of 9 km/min in a locality where $B_{\text{vert.}}$ of the earth's magnetic field is 4×10^{-5} weber/sq m, and $B_{\text{horiz.}} = 2 \times 10^{-5}$ weber/sq m directed due north.

Let it be required to find the potential difference between the ends of the 5-m conductor if its axial length is along the east-west direction.

Either the specified components of the magnetic field may be combined as $\sqrt{B_{\text{vert.}}^2 + B_{\text{horiz.}}^2}$ to find the resultant value of B and then equation (8) may be employed, or the value of $\Delta E/\Delta l$ *due to each component* of **B** may be found separately and then combined to give the resultant value of $\Delta E/\Delta l$. Since, in this case, the horizontal component of **B** coincides in direction with the direction of **v**, this component of **B** is non-productive, and the resultant value of $\Delta E/\Delta l$ is evaluated simply as

$$\frac{\Delta E}{\Delta l} = B_{\text{vert.}}v = (4 \times 10^{-5})\left(\frac{9000}{60}\right) = 0.006 \text{ volt/m} \quad \text{(directed east to west)}$$

and

$$\Delta E = 0.006 \times 5 = 0.030 \text{ volt} \quad \text{(potential difference)}$$

It is left for the reader to apply the right-hand screw rule to **v** and $\mathbf{B}_{\text{vert.}}$ (which is, of course, directed downward) to check the east-to-west direction of the voltage rise which is generated. The "bend" rule may also be applied (say by drawing a cross-sectional view looking from east to west).

In this section, only cases where the entire lengths of conductor (Δl) travel in magnetic fields of uniform density have been considered.

The method may, however, be extended to include variable or non-uniform B by employing space-averaged values of flux density, if necessary, over several small increments of the total length, and then summing the results with due regard for signs. It should be noted in closing this section that, unless the *relative* velocity between Δl and **B** is known, the single-conductor method of calculating generated voltages fails completely.

3. The Loop or Turn Concept. The single-loop law which is stated in equation (6) may be applied to N loops or turns if these turns all experience the same time rate of change of magnetic flux. Equation (6) then reads

$$E_{\text{av.}} = N\frac{\Delta\phi}{\Delta t} \quad \text{or} \quad e = N\frac{d\phi}{dt} \tag{9}$$

Where N is the number of complete turns which experience the same time rate of change of flux, $\Delta\phi/\Delta t$ yields the time-averaged rate of change over a finite interval of time Δt, and $d\phi/dt$ gives the instantaneous time rate of change of flux and hence the instantaneous value of generated voltage.

Faraday discovered this fundamental law about 1833, and equation (9) is often referred to as Faraday's law. Equation (9) is considered by

some people to be more fundamental than equation (8), probably because it was the first to be discovered and is more generally applicable than the single-conductor method of calculating generated voltages.

In order to find the time-averaged value of voltage [$E_{av.}$ of equation (9)] over a finite period of time Δt, it is only necessary to know the *change in flux* that pierces the face of the area bounded by one of the turns. More specifically,

$$E_{av.} = N\frac{\Delta\phi}{\Delta t} = N\frac{(\phi_2 - \phi_1)}{(t_2 - t_1)} \quad \text{(over period } \Delta t) \qquad \text{(9-a)}$$

where ϕ_1 is the magnetic flux which pierces the area bounded by one turn (or the flux linked by one turn) at time t_1

ϕ_2 is the magnetic flux which pierces the same area (or links with one turn) at time t_2.

It should be noted that whereas $E_{av.}$ over a period of time Δt might turn out to be a very small value (or even zero), the *instantaneous generated*

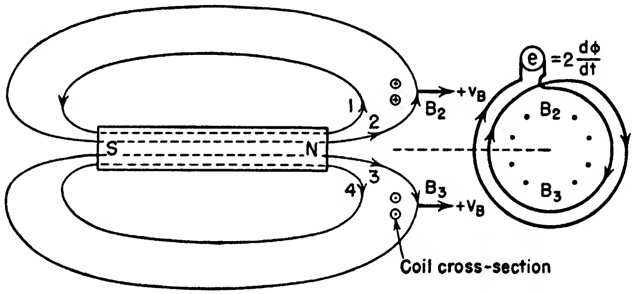


FIG. 3. Magnetic flux piercing the face of a two-turn coil and hence linking with the turns of the coil.

voltages at certain instants during the period Δt might be very large. A simple example will illustrate this point and at the same time make clear what is meant by "flux piercing the area bounded by a turn" and "flux linking with a turn."

In Fig. 3, it will be assumed that the two-turn coil is stationary in the position shown and that the bar magnet is brought up from the left so that the magnetic flux lines emanating from the north pole cut across the conductors as indicated at the B_2 and B_3 positions. Magnetic flux which pierces the face of the coil will either link with the turns of the coil or pierce the same face in the opposite direction in avoiding linkage. The latter type of flux contributes nothing to the net flux piercing the face or to the net flux linkage. The net flux which pierces or crosses the face of the area bounded by one of the turns of a coil times the number

of turns is called flux linkage. Specifically, 1 weber of magnetic flux linking with one turn of conductor is called 1 weber-turn; 1 maxwell of flux linking with one turn is called 1 maxwell-turn.

Returning now to the problem of calculating $E_{av.}$ in Fig. 3, we shall assume that at $t = t_1$ the magnet is far removed (or else so oriented) that no net flux pierces the face of the coil; ϕ_1 in equation (9-a) is zero. In a period of 2 sec, say, the magnet is brought up to the left-hand face of the coil as indicated. During this period of time there is a *change of flux which is directed from left-to-right across this face.* The rule for finding the direction of the generated voltage in loops or turns of this kind is: point the thumb of the right hand in the direction *opposite to the change of flux* which the coil face is experiencing at the moment, and the fingers will encircle the coil in the $+\Delta E/\Delta l$ direction.[2] If, for example, during the period Δt the magnet is brought up to the face of the coil from the left (with no backward motions) the left-hand face of the coil will always experience *a change of flux directed from left to right,* and if the thumb is pointed from right to left to oppose the change-of-flux direction, the fingers of the right hand encircle the coil in such a manner as to say that $+\Delta E/\Delta l$ is directed into the page on the top coil sides and out at the bottom coil sides, as indicated by the $+$ and $\cdot$ symbols respectively.

If, at the position shown in Fig. 3, 0.025 weber of flux links with each of the two turns, then the time-averaged value of voltage over the 2-sec period which has been assumed is

$$E_{av.} = 2\,\frac{(0.025 - 0)}{(2.0 - 0)} = 0.025 \quad \text{volt}$$

At the instant the forward motion of the magnet ceases, the instantaneous voltage reduces to zero (since $d\phi/dt = 0$), and then if the magnet is withdrawn the voltage generated or induced in the coil reverses its circuit direction from that shown in Fig. 3 because the face of the coil is now experiencing a change of flux which is directed from right to left. As shown in Fig. 3, the flux itself may be directed from left to right, and at the same time the change of flux may be oppositely directed owing to the diminution of the amount of flux which is piercing the face of the coil. After a little thought has been given to the meaning of *change-of-flux direction* the rule given above is very easily applied.

[2] This rule is an application of *Lenz's law* which states that the instantaneous voltage generated or induced in an electric circuit by a time-varying magnetic flux acts in such a direction around the circuit as to *oppose the change of flux* which is taking place (at the moment) across the face of the turns of the circuit.

Example 1. Let it be required to find the time-averaged voltage generated in the two-turn coil of Fig. 3 if the magnet (assumed to possess cylindrical symmetry) is revolved about its central N-S axis.

Since, in this case, there is *zero change of flux linkage* (or no change in flux which pierces the face of the coil) the instantaneous, as well as the time-averaged value of voltage, is zero.

Example 2. Let it be required to find the time-averaged voltage generated in the coil of Fig. 3 over a period of $\Delta t = 0.05$ sec if the magnet is revolved about an axis which is normal to the plane of the page and centrally located between the north and south poles. The speed is 10 rps, and time is to be counted from the instant the magnet is in the position shown in Fig. 3, the flux linking with a single turn in this position being taken as +0.025 weber.

At $t = 0 = t_1$,

$\phi_1 = +0.025$ weber (directed left to right through coil face)

At $t = 0.05 = t_2$,

$\phi_2 = -0.025$ weber (directed right to left through coil face)

The value $\phi_2 = -0.025$ weber is obtained from the fact that at $t = 0.05$ sec the magnet has turned one-half revolution, thus placing the south pole in the position directly in front of the left-hand coil face.

From equation (9-a),

$$E_{\text{av.}} = 2\,\frac{(-0.025 - 0.025)}{(0.05 - 0)} = -2.0 \quad \text{volts}$$

where the minus sign indicates only that the circuit direction of the voltage generated during this period of 0.05 sec is directed oppositely to that shown in Fig. 3 which was obtained with change of flux directed from left to right. In the period here considered the change-of-flux direction has always been from right to left, first because of the removal of the north pole and then because of the appearance of the south pole.

4. Time Rates of Change-of-Flux Linkage. The concept of magnetic flux lines linking with a single turn, or with N tightly packed turns, has been referred to in the preceding section. The *linking* of magnetic flux with turns of an electrical circuit is in the same sense that neighboring elements of a chain link with one another. Flux linkage ($N\phi$) as a concept is of great importance, not only in the calculation of voltages but, as will be shown later, in the calculation of inductance or electrical inertial effects.

If the total flux linkage $\Sigma(N\phi)$ of the electrical circuit is known as a function of time or if it can be evaluated in some manner, the voltage generated in the electrical circuit, due to ($N\phi$) is simply the time rate of change of ($N\phi$). In order to account for the fact that one portion of the electrical circuit (say N_1 turns) might be linked with a flux (say ϕ_1)

which is different from the flux linking some other portion of the circuit, we write equation (9) as

$$E_{\text{av.}} = -\frac{\Delta(N\phi)}{\Delta t} \quad \text{or} \quad e = -\frac{d(N\phi)}{dt} \tag{9-b}$$[3]

where $N\phi$ is understood to be the summation of all the flux linkages (with due regard for sign) of the entire circuit which is being investigated; that is,

$$(N\phi) = \Sigma(N_1\phi_1 + N_2\phi_2 + \cdots) \tag{10}$$

Then each turn or each group of turns linking with the same flux may be considered separately if the occasion demands.

An electrical circuit might be formed of the two coils (N_1 and N_2) shown in Fig. 4. This may be done by connecting terminal $1'$ of the N_1 coil to terminal $2'$ of the N_2 coil and then viewing terminals 1 and 2 as the terminals of the electrical circuit which is under investigation. Or terminal $1'$ might be connected to terminal 2 and the circuit between terminals 1 and $2'$ investigated. It might be anticipated that the voltage developed between terminals 1 and 2 (with $1'$ connected to $2'$) due to time rates of change of $(N_1\phi_1)$ and $(N_2\phi_2)$ would be different from the voltage developed between terminals 1 and $2'$ (with $1'$ connected to 2). The details are outlined in the following example, where it is assumed that the mmf coil of Fig. 4 establishes sinusoidally time varying fluxes ϕ_1 and ϕ_2 which link respectively with N_1 and N_2 turns.

Example. In Fig. 4, let

$$\phi_1 = 0.04 \sin (377t) \quad \text{weber}$$

and

$$\phi_2 = 0.03 \sin (377t) \quad \text{weber}$$

These equations mean that the exciting (or mmf) coil establishes a maximum flux of 0.04 weber through the N_1 turns and a maximum flux of 0.03 weber through the N_2 turns, and that the time variation of these fluxes is sinusoidal at an angular frequency of 377 radians/sec. Since angular frequency ω is 2π times the frequency in cycles per second, an angular frequency of 377 radians/sec corresponds to 60 cycles/sec. Thus the exciting coil of Fig. 4

[3] The minus sign is employed in connection with these equations simply to warn the reader that the voltage *rises* calculated by these equations are *oppositely* directed to the fingers of the right hand when the thumb is pointed in the *direction of the change of flux*. In later courses where wave propagation is studied, this minus sign takes on somewhat more significance than it does here, where we simply calculate the magnitude of the average or instantaneous voltage and then apply the right-hand rule to determine the direction around the circuit in which the voltage *rise* takes place.

connected to an ordinary 60-cycle supply voltage is capable of establishing the specified values of ϕ_1 and ϕ_2.

$$(N_1\phi_1) = 3 \times 0.04 \sin (377t) = 0.12 \sin (377t) \quad \text{weber-turn}$$

$$(N_2\phi_2) = 2 \times 0.03 \sin (377t) = 0.06 \sin (377t) \quad \text{weber-turn}$$

During the first quarter of a cycle (after $t = 0$ in the above equations) it will be assumed that the flux increases from zero, as demanded by the sine terms at $t = 0$, to maximum values which are directed downward through the coils N_1 and N_2 as shown in Fig. 4 which, of course, only pictures instantaneous directions of flux and circuit directions. The maximum values that ϕ_1 and ϕ_2 can attain have already been specified as 0.04 and 0.03 weber respectively.

If terminal 1′ of coil 1 is connected to terminal 2′ of coil 2, the problem of finding the instantaneous voltage between terminals 1 and 2 of the combined circuit is simply that of finding $e_{11'}$ and $e_{2'2}$ and combining $e_{11'}$ and $e_{2'2}$ to give

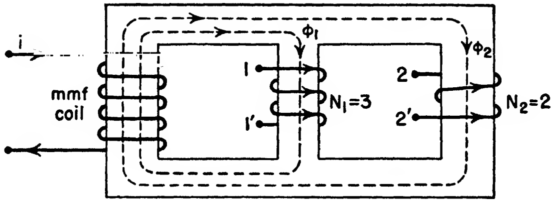


FIG. 4. Illustrating $(N_1\phi_1)$ and $(N_2\phi_2)$. The voltage rise directions shown on N_1 and N_2 apply for a change-of-flux direction which is downward through the faces of N_1 and N_2.

the voltage rise from terminal 1 to terminal 2. Thus, by using the instantaneous form of equaton (9-b), we find

$$e_{11'} = -\frac{d}{dt}[0.12 \sin (377t)] = -45.24 \cos (377t) \quad \text{volts}$$

$$e_{2'2} = -\frac{d}{dt}[0.06 \sin (377t)] = -22.62 \cos (377t) \quad \text{volts}$$

Combining these two results yields

$$e_{12} = e_{11'} + e_{2'2} = -67.86 \cos (377t) \quad \text{volts} \quad \text{(for additive connection)}$$

The minus sign signifies that this voltage is opposite in direction to the direction of the fingers of the right hand when the thumb is pointed in the direction of the change of flux. During the first quarter-cycle after $t = 0$ the flux is directed downward in Fig. 4 and increasing in magnitude in this direction, therefore the change-of-flux direction through the coils is downward. An application of the right-hand rule will show that the voltage rise during this period is from terminal 1 to terminal 2 as shown in Fig. 4.

If terminal 1′ had been connected to terminal 2, a study of the instantaneous circuit directions will show that

$$e_{12'} = e_{11'} + e_{22'} = e_{11'} - e_{2'2} = -22.62 \cos (377t) \quad \text{volts}$$

and with this type of connection the coils are said to be connected subtractively. In this particular case the maximum magnitude of the generated or induced voltage is only one-third as large with the subtractive connection as it is with the additive connection.

In a-c cases of the kind considered here, the investigation of the relative polarities of the voltages need be made at only one instant of time. The voltages actually go through cyclic variations (as expressed by the cosine terms in this case) with alternations which are first in one direction around the circuit and then in the other but always with the same *relative* polarities.

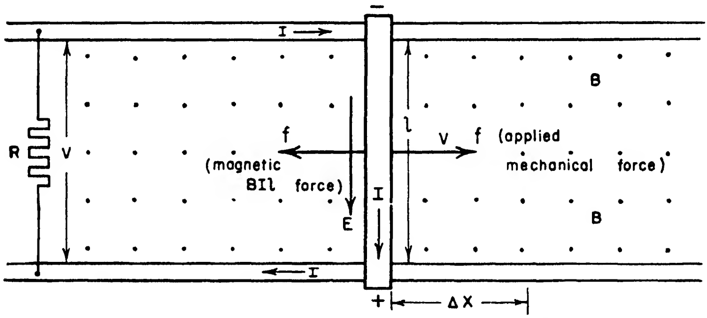


FIG. 5. An elementary linear velocity electrical generator.

5. Energy and Power Transformations in a Linear Velocity Generator. It will be assumed that the moving armature conductor of the generator shown in Fig. 5 is capable of linear motion along the horizontal rails and that a driving force is applied to the armature conductor which moves it a distance Δx to the right in time Δt. Then

By single-conductor concept:

$$\frac{\Delta E}{\Delta l} = Bv \quad \text{or} \quad E = Blv \tag{11}$$

By single-loop concept:

$$E_{\text{av.}} = \frac{\Delta \phi}{\Delta t} = \frac{(Bl\,\Delta x)}{\Delta t} = Blv \tag{12}$$

where v is the time-averaged value of the velocity, $\Delta x/\Delta t$.

The direction of the voltage rise E in the armature conductor may be obtained by any one of several rules, for example, by: (1) thinking of turning vector $\mathbf{v}$ out of the page 90° (to coincide with the direction of $\mathbf{B}$)

to obtain $+E/l$ downward in the conductor; (2) pointing the thumb of the right hand downward to oppose the change-of-flux direction (which is upward through the loop formed by the moving conductor, the rails, and the resistance R when the conductor is moved to the right in Fig. 5), thus obtaining $+E$ in the clockwise direction around the closed loop.

The generated voltage E drives a current around the loop as indicated and, if it is assumed that neither the armature conductor nor the rails have an appreciable resistance compared to resistance R, the electrical energy supplied to the load R in time Δt is

$$W_{\text{el.}} = EI\,\Delta t = RI^2\,\Delta t \tag{13}$$

and the time-averaged value of power delivered to the load over this period of time is

$$P_{\text{el.}} = \frac{W_{\text{el.}}}{\Delta t} = RI^2 \tag{14}$$

Let us see what we pay for the electrical energy which is generated and then transformed into heat energy in terms of mechanical energy supplied to the generator. It is plain that the applied force must at least equal the BIl force in the armature conductor. This backwardly directed magnetic force is inevitably present in any electromagnetic generator which is delivering current. All frictional and other losses being neglected, the least mechanical energy which the prime mover would be required to supply during the period Δt would be

$$W_{\text{mech.}} = f\,\Delta x = (BIl)\,\Delta x = EI\,\Delta t \tag{15}$$

The fact that the mechanical energy supplied by the prime mover must at least equal $EI\,\Delta t$ follows from the law of the conservation of energy as well as from the mathematical equivalence of $(BIl)\,\Delta x$ and $(Blv)\,I\Delta t$.

The time-averaged value at which the prime mover must deliver power over the interval of time Δt is plainly.

$$P_{\text{mech.}} = \frac{W_{\text{mech.}}}{\Delta t} = EI \tag{16}$$

where $P_{\text{mech.}}$ is expressed in watts if E and I are expressed in volts and amperes respectively.

Energy taken from an electrical system is paid for in terms of mechanical energy delivered to the mechanical-electrical system by the prime mover. Every time an incandescent lamp is switched on, for example, an additional BIl force appears at the armature conductors of the generator (or generators) which will tend to slow down the speed of

the generator (thereby reducing the voltage E) unless this added BIl force is met with an additional driving force. This additional driving force can come only from the prime mover.

6. The Spinning-Coil Type of A-C Generator. The two-turn rectangular coil shown in Fig. 6 is considered to be located in a region where the magnetic flux density is uniformly distributed and directed as shown. The power required to maintain this magnetic field through the air medium (in which the coil rotates) would probably be greater than the power output of the generator, but in spite of this we shall investigate the possibilities of the spinning coil as a source of electrical energy. (A more practical type of a-c generator is shown in Fig. 11 which is used in connection with Prob. 20. See page 278.)

Choosing the reference position of the two-turn coil of Fig. 6 as that which is labeled $t = 0$, we note that in this reference position maximum

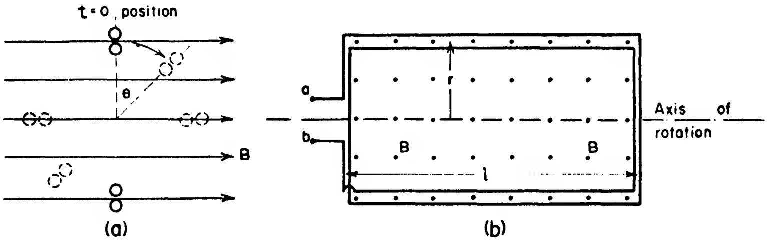


FIG. 6. A two-turn coil rotating in a uniformly distributed magnetic field.

flux linkage occurs; the **B** vectors shown ultimately close on themselves and thereby link with the turns of the coil. We also note that in the reference position the coil sides are traveling in the same direction as the **B** vectors, and hence zero voltage will be generated at $t = 0$. The other way of looking at the situation is to note that no *change* of flux linkage is taking place at $t = 0$ even though the coil face is pierced by the maximum amount of flux possible during a revolution. In the reference position,

$$(N\phi)_{\text{max.}} = NBA = NB(2lr) \tag{17}$$

At any instant of time after $t = 0$ when the coil is displaced by an angle θ from the reference position it is plain that the flux linkage is

$$(N\phi)_\theta = NBA_\theta = NB(2lr \cos \theta) \tag{18}$$

where A_θ is the area of the coil at angle θ which is presented to the direction of the flux and through which flux linkages can occur.

If the coil is spinning or rotating about the axis shown in Fig. 6 at a constant speed S, expressed in revolutions per second,

$$\theta = \frac{\text{radians}}{\text{sec}} \times \text{sec} = \omega t \quad \text{radians}$$

and θ may be expressed in terms S and as a function of time t:

$$\theta = \frac{2\pi}{T} t = 2\pi S t \quad \text{radians} \tag{19}$$

where T is the period of one revolution and S is the revolutions per second.

Thus, from equation (18), $(N\phi)_\theta$ may be expressed as a function of time:

$$(N\phi)_t = 2rlNB \cos (2\pi St) \quad \text{flux linkages} \tag{20}$$

The time rate of change of these flux linkages is

$$e = -\frac{d(N\phi)}{dt} = (2\pi S)(2rlNB) \sin (2\pi St)$$

or

$$e = 4\pi rlSNB \sin (2\pi St) \quad \text{volts} \quad \text{(in mks units)} \tag{21}$$

Equation (21) informs us that the instantaneous voltage which is generated by the revolving coil is directly proportional to the coil area, the speed S, the number of turns N, and the magnitude of the flux density B; also that the time variation of the voltage is sinusoidal, the frequency of this time variation being equal (in cycles per second) to the speed S expressed in revolutions per second.

It is only necessary to check the direction of voltage rise around the loop at an instant shortly after the coil leaves the $t = 0$ position. During the first quarter revolution after $t = 0$, the change-of-flux direction is from right to left in Fig. 6-a, since the flux linkage which was maximum and positive at $t = 0$ is reduced to zero during this first quarter of a revolution. The application of the right-hand rule will show that the voltage rise around the coil during this period is "out" at the upper coils sides and "in" at the lower coil sides; in other words, the voltage rise is from b to a. Thus e in equation (21) means e_{ba} or voltage rise from b to a; and the sine term in equation (21) will automatically tell us when e_{ba} is negative. Physically, terminal a in Fig. 6-b is actually positive relative to terminal b during the first half-revolution of the coil.

Example 1. *Half-Revolution Average Voltage Calculation.* Let it be required to find the time-averaged voltage generated in the 2-turn coil shown in Fig. 6

during the period of time in which the coil rotates one-half revolution from the $t = 0$ position, under the following conditions:

Uniform speed of 1200 rpm, $B = 200$ maxwells/sq cm or $B = 0.02$ weber/sq m

$$l = 30 \text{ cm}$$

$$r = 10 \text{ cm}$$

The calculations may be made either in terms of changes of flux linkage or by the single-conductor method, if it recognized that, in the latter, the four active 30-cm coil sides are in additive series between terminals b and a. By the single-conductor method,

$$E_{\text{av.}} = N_c B l v_{\text{av.}}$$

where N_c represents the four single conductors and $v_{\text{av.}}$ is the time-averaged velocity of the conductors which is *normal* to the direction of **B**, since only the component of **v** which is normal to **B** is effective in producing voltage. As applied to Fig. 6-a, the horizontal component of the velocity may be ignored since it is parallel to the **B** vectors:

$$v_{\text{av.}} = \frac{2r}{\frac{T}{2}} = \frac{0.2}{\frac{1}{40}} = 8 \text{ m/sec} \quad \left(\text{over } \Delta t = \frac{T}{2} \text{ sec}\right)$$

Hence

$$E_{\text{av.}} = (4)(0.02)(0.3)(8) = 0.192 \text{ volt} \quad \left(\text{over } \Delta t = \tfrac{1}{40} \text{ sec}\right)$$

which may readily be shown to be directed from terminal b to terminal a in Fig. 6-b. The above result may be easily checked by the $N\ (\Delta\phi/\Delta t)$ method of calculation.

Example 2. *Instantaneous Voltages.* Equation (21) defines the voltage rise from b to a at any instant of time after $t = 0$. The maximum value of voltage that is generated at any one instant of time in the coil shown in Fig. 6 is, for the conditions specified in Example 1,

$$E_{\text{max.}} = 4\pi r l S N B = (4\pi)(0.1)(0.3)(20)(2)(0.02) = 0.096\pi = 0.3016 \quad \text{volt}$$

or $2/\pi$ times the time-averaged voltage over one-half revolution.

Since equation (21) is derived on the flux-linkage basis, it is plain that N in the above expression for $E_{\text{max.}}$ refers to *turns*, not single conductors.

7. An Elementary Type of D-C Generator. In d-c generators, little attention need be paid to the instantaneous voltage which is generated in any particular conductor, because sliding contacts render all conductors which are facing a pole (or group of like poles) in additive series with respect to the brushes or terminals of the machine. The manner in which this is accomplished is illustrated in Fig. 7. It will be observed in Fig. 7 that two current paths exist which are in parallel between the

brushes. In the actual machine the entire surface would be covered with active conductors but only four are shown in Fig. 7 in order to simplify the drawing to a point where it can readily be understood.

Current coming into the lower brush in Fig. 7 divides at the point of sliding contact (upon leaving the brush and entering the generator winding), one-half of it going to the upper brush by way of the active conductors facing the north pole; the other half by way of the active conductors facing the south pole. Current flows in the direction shown in Fig. 7 when a load is connected to the terminals because, for the direction of rotation shown, there is developed a voltage rise *from* the lower (or negative) terminal *to* the upper (or positive) terminal in each of the two parallel paths. Thus there are two paths in which voltages are developed in this machine, and the two generated voltages are in parallel. In determining the voltage developed at the terminals of the generator only the voltage developed in one path need be determined, since by symmetry the other voltage is identical in magnitude and direction relative to the terminals of the machine.

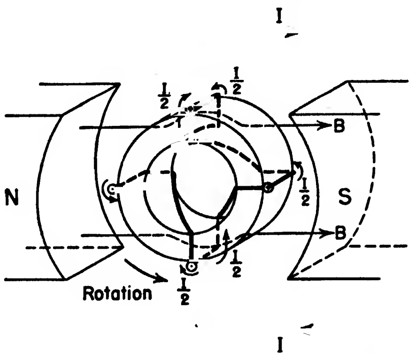


FIG. 7. An elementary type of d-c generator illustrating the two parallel armature paths.

The straight conductors on the outside surface of the rotating armature are so placed that their velocities will be always normal to the magnetic field which is shown as following the iron core of the armature. Actually the air gap between the pole faces and the rotating armature is very short, being only a few hundredths of an inch in small machines. On a single-conductor basis, only the straight portion of the armature winding which is on the surface of the iron core is effective in generating voltage; since the "inner" conductors do not cut across the magnetic field.

The time-averaged voltage developed by a d-c machine is essentially the same as the instantaneous generated voltage if a sufficiently large number of active armature conductors is employed. On small machines where the number of active conductors is likely to be small, there is often a significant fluctuation of the instantaneous voltage about the time-averaged value. This fluctuation is referred to as *commutator ripple*, since the sliding contact arrangement used on modern machines is called a commutator.

The calculations are based on the voltage generated in the active armature conductors as they cut across the magnetic field, and the time-averaged voltage developed by a two-pole d-c generator is simply

$$E_{\text{av.}} = N_c B_{\text{av.}} lv = N_c B_{\text{av.}} l(2\pi r S) \tag{22}$$

where N_c is the number of active armature conductors in *one* path
$B_{\text{av.}}$ is the space-averaged value of flux density brush to brush
l is the axial length of the active armature conductors
r is the radius at which the active conductors are from the axis of rotation
S is the speed of rotation expressed in revolutions per second.

If the above information is available or can be evaluated in some manner, the problem of finding the voltage developed by this type of generator resolves itself into putting numbers into equation (22).

Example. Refer to the cross-sectional view of the d-c machine shown in Fig. 10, page 237. This same machine has been used to illustrate d-c motor calculations, and the data specified have been

Voltage applied to the brushes, $V = 36$ volts (motor voltage)
Total motor armature current (I in Fig. 10-a) = 20 amp
Total number of armature conductors, $N = 24$
Axial length of each conductor, $l = 20$ cm
Radius at which each armature conductor is from axis of rotation, $r = 10$ cm
Space distribution of flux density as shown in Fig. 10-b, page 237.
Speed of rotation as a motor, 3000 rpm.

Considering this same machine from a generator point of view we find 12 active conductors in series between brushes which are cutting magnetic flux, the space-averaged density brush-to-brush being 0.395 weber/sq m as previously evaluated. The voltage generated in these motor conductors, brush-to-brush, has a magnitude which is given by equation (22). Thus

$$E_{\text{av.}} = N_c B_{\text{av.}} l(2\pi r S) = (12)(0.395)(0.2)(6.28 \times 0.1 \times 50) = 29.76 \quad \text{volts}$$

The machine is running as a motor with 36 volts applied brush-to-brush which drives it in the counterclockwise direction, as any of the *force* rules will show. Whereas the motor current is directed "in" in the conductors facing the north pole of Fig. 10-a, page 237, the generated voltage is directed "out" in these same conductors, as any of the generator rules will show. For this reason the generated voltage in a motor is called *back emf.*

The *back emf* of a motor is less than the applied voltage V by the RI voltage drop in the armature winding. In the present case

$$RI = 36 - 30 = 6.0 \quad \text{volts} \quad \text{(to two-significant-figure accuracy)}$$

Since the motor current I under this condition of operation has been specified

as 20 amp, the resistance of the armature brush-to-brush may be evaluated as

$$R_{\text{arm.}} = \frac{6.0}{20} = 0.3 \quad \text{ohm}$$

Since this value of resistance represents the resistance of two equal paths in parallel, the resistance of each path is 0.6 ohm.

8. Governor Action of Back Emf. As shown in the preceding example the generated voltage in the armature windings of a d-c machine operating as a motor is

$$E_{\text{back}} = V - R_{\text{arm.}}I \tag{23}$$

where V is the voltage applied to the motor terminals
$R_{\text{arm.}}$ is the resistance, brush-to-brush, of the armature windings
I is the total armature current which drives the motor.

The voltage which is generated in the armature winding of any electromagnetic motor opposes the motor current and in so doing provides the motor with a *valve* which will automatically produce more or less driving torque as demanded by the connected load.

This automatic valve or governor action is explained simply in terms of the back emf. For increase of load, the sequence of the governor action is:

(*a*) The increased load tends to slow the speed of the machine.

(*b*) The slower speed lowers the back emf which varies as the speed.

(*c*) The lower back emf (E) permits more current, $I = (V - E)/R_{\text{arm.}}$ to flow.

(*d*) The increase in I provides increase in driving force, $f = BIl$.

The sequence reverses when the load is lessened so that electric motors, generally speaking, need no external governors as do most other types of motors. At no load the speed increases to a value which makes the *back emf* essentially equal to the voltage applied to the motor terminals, thereby reducing the current to a very small value.

The product of the *back emf* and the armature current is a direct measure of the amount of electrical power which is transformed into mechanical power. Since the electrical power supplied to the motor terminals (VI) is transformed partly into mechanical power ($E_{\text{back}}I$) and partly into heat power which is wasted in the resistance of the armature windings (RI^2), it follows from the law of conservation of energy (or power) that

$$\underset{\text{(input)}}{VI} = \underset{\text{(mechanical)}}{E_{\text{back}}I} + \underset{\text{(heat)}}{RI^2} \tag{24}$$

Thus it is shown that the generated voltage within a motor winding (E_{back}) is a controlling factor in the operation of the motor and, without

a knowledge of this factor, the performance of the motor cannot be theoretically predicted.

Example. In the last example, a motor (supplied with 20 amp at 36 volts) was shown to develop a back voltage of 30 volts, to two-significant-figure accuracy, and to have an armature resistance of 0.3 ohm. Under this condition of operation,

$$\text{power input} = VI = 36 \times 20 = 720 \quad \text{watts}$$

$$P_{\text{mech.}} = E_{\text{back}} I = 30 \times 20 = 600 \quad \text{watts}$$

$$\text{heat power} = RI^2 = 0.3 \times 20^2 = 720 - 600 = 120 \quad \text{watts}$$

If the connected load is lessened to a point where only 300 watts of mechanical power are required, the motor automatically adjusts itself to meet the condition demanded by equation (24) which, for 300 watts of mechanical power and 36 volts applied to the terminals, is

$$36I = 300 + 0.3I^2 \qquad \text{or} \qquad I^2 - 120I + 1000 = 0$$

This equation may be solved for I to show that the armature current is *reduced* to about 9.0 amp (from the 20-amp value) when the mechanical load is reduced to 300 watts (from the previous 600-watt value).

In solving the above equation for I, it has been assumed that I decreases for a decrease in connected load, as would be the case physically if the motor were operating on that part of its speed-torque curve where less torque is accompanied by higher speed.

Under the lighter load condition ($P_{\text{mech.}} = 300$ watts),

$$E_{\text{back}} = V - R_{\text{arm.}} I = 36 - 0.3 \times 9.0 = 33.3 \quad \text{volts}$$

and, since E_{back} varies directly with speed (assuming constant flux density),

$$\text{speed under reduced load} = \frac{33.3}{30} \times 3000 = 3330 \quad \text{rpm}$$

In the actual machine the armature current affects to some extent the resultant flux density in the air gap of the machine, but this effect (known as armature reaction) will not be considered here since it is a subject which is dealt with in detail in electrical machinery texts.

9. Summary. The potential gradient along the axial length Δl of a single straight conductor moving at velocity v relative to a magnetic field is

$$\frac{\Delta E}{\Delta l} = Bv \sin \theta \Big]_{\mathbf{v}}^{\mathbf{B}} = B_n v = B v_n \tag{25}$$

where $B_n = B \sin \theta \Big]_{\mathbf{v}}^{\mathbf{B}}$ and $v_n = v \sin \theta \Big]_{\mathbf{v}}^{\mathbf{B}}$.

Equation (25) is simply a "finite increment" form of the more general law

$$\left.\frac{dE}{dl}\right]_{\text{max.}} = -\mathcal{E} = Bv_n \underline{/\text{directed normal to the } \mathbf{Bv} \text{ plane}} \tag{26}$$

and it is by way of this equation that the electric field $\mathcal{E}$ and the magnetic field **B** are linked together in later courses to explain radio wave propagation, a subject which is well beyond the scope of a first course.

The direction of the potential gradient $(\Delta E/\Delta l)$ or $(dE/dl)_{\text{max.}}$ is obtained from the right-hand screw rule by thinking or rotating the **v** vector (*of the moving conductor*) through the smaller angle to the **B** direction, the resulting right-hand screw motion yielding the direction of $+(\Delta E/\Delta l)$ which is normal to the plane in which **B** and **v** are located.

The voltage generated or induced in an electrical *circuit* is

$$E_{\text{av.}} = -\frac{\Delta(N\phi)}{\Delta t} \quad \text{or} \quad e = -\frac{d(N\phi)}{dt} \tag{27}$$

where $(N\phi)$ represents the resultant flux linkages of the magnetic field with the turns of the electrical circuit. Equation (27) represents a fundamental law which states that the *voltage generated in an electrical circuit* by a magnetic field *is the time rate of change of flux linkages* which occur within the circuit.

The direction of the voltage rise around the electrical circuit at any instant of time is obtained by the right-hand rule, the thumb pointing opposite to the change-of-flux direction which is occurring at that instant through the circuit and the fingers of the right-hand pointing in the direction of the voltage rise.

Since relative motion is involved in equation (25) and since some care must be exercised in visualizing all possible flux linkages in equation (27), the evaluation of magnetically generated voltages may become an interesting problem where neither Bv nor $(N\phi)$ can be readily visualized.

PROBLEMS

1. Refer to Fig. 5, page 264. Find the magnitude of the voltage generated in the moving armature conductor of this "linear-velocity" generator if $l = 1$ m, the velocity (directed as shown) is uniform at 10 m/sec, and the magnitude of the flux density (uniformly distributed and directed as shown) is 0.22 weber/sq m.

Check the direction of the generated voltage E shown in Fig. 5 by both the right-hand screw rule and the "bend-of-flux" rule shown in Fig. 2.

2. The axle of a locomotive truck is 4 ft 8.5 in. (or 1.435 m) in length. The locomotive is moving due south at a speed of 120 km/hr in a region where the earth's magnetic flux density is $\sqrt{20} \times 10^{-5}$ weber/sq m directed downward from a hori-

zontal plane with an angle of 63.45°, the horizontal component being directed due north.

(*a*) Find the current which flows momentarily through the loop formed by the axle, the wheels, the rails, and a switch tie-rod (which joins the rails at the bottom) if the resistance of this loop is 0.01 ohm. Base calculations on $I = E/R$.

(*b*) Which end of the axle is the positive terminal of the generator?

3. A straight conductor 1 m in length moves northwest (45° from due north) in a horizontal plane at 20 m/sec in a region where $B_{\text{vert.}}$ of the earth's magnetic field is 4×10^{-5} weber/sq m; and $B_{\text{horiz.}}$ is 2×10^{-5} weber/sq m directed due north.

What is the potential difference between the ends of the conductor if the axial length of the conductor remains oriented in the N-S direction?

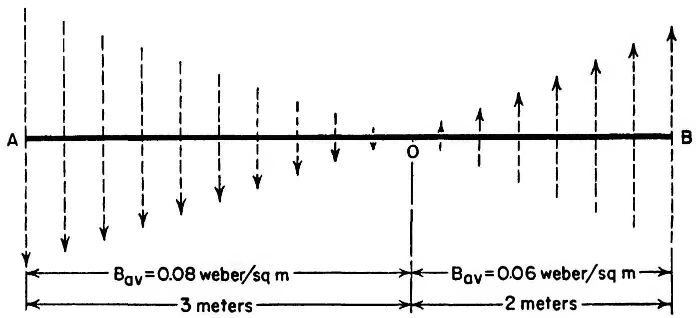


FIG. 8. See Prob. 5.

4. The horizontal conductor (1 m in length) of Prob. 3 is assumed to rotate in a horizontal plane about a vertical axis which is located at one end of the rod, the speed of rotation being 10 rps.

What is the time-averaged value of voltage difference between the ends of the conductor?

5. The straight conductor shown in Fig. 8 is moving in a horizontal plane with its velocity directed at right angles to the plane of (and into) the page. The flux density distribution over the axial length of the 5-m conductor is as shown in Fig. 8, and only the vertical components of B which are shown are present.

What is the electric potential of the B end of the conductor relative to the A end if the velocity is 50 m/sec directed inwardly as specified?

6. A straight metal rod 1 m long rotates about an axis which is located at one end of the rod, the axis of rotation being perpendicular to the axial length of the rod. The entire area swept over by the rod possesses a uniformly distributed magnetic flux density of 0.01 weber/sq m directed at right angles to the axial length of the rod. Find the potential difference between the ends of the rod when the speed of rotation is 600 rpm.

7. A magnetic flux of 10^7 maxwells (or 0.1 weber) which initially links with a 200-turn coil is reduced to zero in 0.5 sec.

What is the time-averaged voltage generated in this coil during the 0.5-sec period?

8. A coil of 1000 turns encircles the mid-section of a bar magnet, and in this position it is known that each turn is linked with 2510 maxwells of flux. The coil is stationary and the magnet free to move. If by some means the bar magnet is ejected from the neighborhood of the coil in a period of 1 sec, find the time-averaged voltage generated in the coil during the 1-sec period.

9. In Fig. 3, page 259, assume that the bar magnet is being moved up to the face of the coil so that the B vectors (in the process of linking with the turns of the coil) cut across the conductors with velocities v_B as shown.

Check the direction of the generated voltage around the electrical circuit shown in Fig. 3 to be "in" at the top and "out" at the bottom by: (*a*) the right-hand rule which points the thumb opposite to the change-of-flux direction; (*b*) the "bend-of-flux" rule, thinking of the "bend" which either B_2 or B_3 suffers just before "snapping" into the linkage position shown in Fig. 3; (*c*) the right-hand screw rule where the *conductor velocity* vector is turned to coincide with the **B** direction.

10. Assume that a 100-turn coil encircles the mid-section of a cylindrical shaped permanent magnet, the cross-sectional area of which is 2 sq cm. It is known that the time-averaged voltage during the 0.5-sec period required to eject the coil from the vicinity of the magnet (say by stretched rubber bands) is 4 mv.

(*a*) What is the flux density over the 2-sq-cm cross-sectional area of the magnet?

(*b*) What flux emanates from the entire north pole region of the magnet?

11. Refer to Fig. 4, page 263, where it will be assumed that the current in the mmf coil is so controlled that ϕ_1 expressed as a function of time is

$$\phi_1 = 9.6t \quad \text{weber,} \qquad \text{for } t \text{ in seconds,} \qquad \text{and} \qquad 0 < t < \tfrac{1}{240} \text{ sec}$$

At the end of the $\frac{1}{240}$-sec period, the exciting current attains a constant value after which $\phi_1 = 0.04$ weber (constant).

(*a*) What is the instantaneous voltage generated in N_1 (of 3 turns) at $t = 0.002$ sec?

(*b*) What is the time-averaged voltage between terminals $1'$ and 1 of the N_1 coil during the first $\frac{1}{240}$ sec after $t = 0$?

(*c*) What is the instantaneous voltage between terminals $1'$ and 1 at $t = 0.005$ sec?

12. Refer to Fig. 4, page 263, where it will be assumed that the current in the mmf coil is so controlled that ϕ_1 changes from zero (at $t = 0$) to 0.04 weber at $t = \frac{1}{240}$ sec linearly with respect to time. During this period ϕ_1 is directed downward as shown in Fig. 4, as is ϕ_2 which changes from zero (at $t = 0$) to 0.03 weber at $t = \frac{1}{240}$ sec linearly. If terminal $1'$ of the 3-turn coil is connected to terminal $2'$ of the 2-turn coil, find

(*a*) the instantaneous voltage between terminals 1 and 2 at $t = \frac{1}{480}$ sec;

(*b*) the time-averaged voltage between terminals 1 and 2 during the $\frac{1}{240}$-sec period between $t = 0$ and $t = \frac{1}{240}$ sec;

(*c*) the effective value of the voltage between terminals 1 and 2 if it is assumed that the flux variations are repeated cyclically each $\frac{1}{240}$ sec.

13. Refer to Fig. 9. A portion of the flux established in the surrounding neighborhood by the $N_{exc.}$ coil when switch S is closed is shown to link with a nearby coil. In a 0.2-sec period after switch S is closed the $N_{exc.}$ coil establishes 0.5 weber of flux

(through its center section) of which 10 per cent is assumed to link with the nearby or "coupled" circuit.

(*a*) During the 0.2-sec interval required for the establishment of the flux, what is the change-of-flux direction through the coupled turns, up or down in Fig. 9, and in what direction is the generated voltage rise in the coupled circuit, clockwise or counterclockwise?

(*b*) What is the time-averaged value of voltage generated in the coupled circuit over the 0.2-sec period, if it is assumed that the effective number of "linking" turns of the coupled circuit is 4?

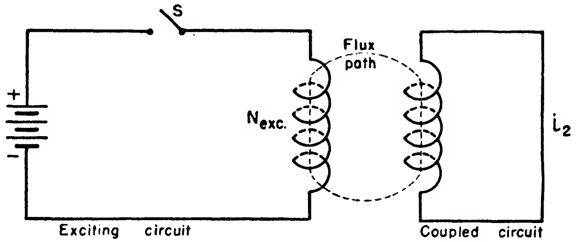


FIG. 9. See Prob. 13.

(*c*) When switch S is opened, does i_2 flow clockwise or counterclockwise around the coupled circuit?

14. Refer to Fig. 6, page 266. If $B = 200$ maxwells/sq cm, $l = 30$ cm, $r = 10$ cm, and the speed of rotation is 20 rps, find the time-averaged voltage generated in the 2-turn coil over the 0.02-sec period which elapses between the time when the coil is in the $t = 0$ position and 0.02 sec later.

15. Refer to Fig. 6, page 266, and assume that the axis of rotation is changed from the central position shown in Fig. 6-b to the lower coil-side position shown in that figure. Assume also that the two conductors which form the lower coil side are of such small diameter that they practically occupy this new axis of rotation. The known data are: $l = 30$ cm, $2r = 20$ cm, $B = 0.02$ weber/sq m, rpm = 1200.

(*a*) Determine the time-averaged voltage generated in the two-turn coil during the period of time required to complete one-half revolution starting from the $t = 0$ position shown in Fig. 6-a.

(*b*) Show that the expression for e_{ba} as a function of time under this condition of operation is

$$e_{ba} = 0.3016 \sin (40\pi t) \quad \text{volt}$$

16. Assume that the elementary generator-motor combination shown in Fig. 10 operates under the following conditions with the generator armature conductor moving to the right, as indicated, under the influence of a prime mover.

Generator Data	*Motor Data*
$R_{\text{gen.}} = 0.010$ ohm	$R_m = 0.0156$ ohm
$B_{\text{gen.}} = 6450$ maxwells/sq in.	$B_m = 3225$ maxwells/sq in.
$l_{\text{gen.}} = 31.5$ in.	$l_m = 63$ in.
$v_{\text{gen.}} = 105$ ft/sec	$v_m = 52.5$ ft/sec

The combined series resistance of the armature conductors, namely, 0.0256 ohm, is assumed to be the only resistance in the electrical circuit, and the flux densities are assumed to be uniformly distributed and directed as shown, "out" in the case of the generator and "in" in the case of the motor.

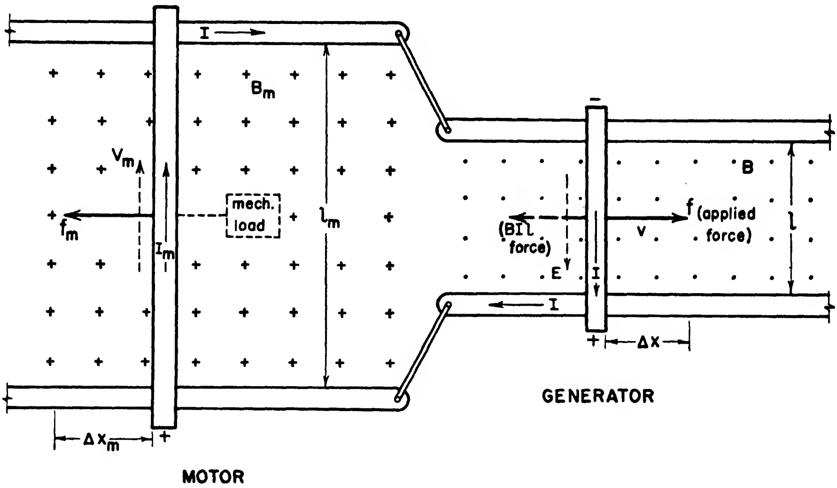


FIG. 10. A single-conductor generator connected to a single-conductor motor. See Probs. 16, 17, 18, and 19.

(*a*) Find the internally generated voltage in the generator conductor.
(*b*) Find the *back* voltage generated in the motor conductor.
(*c*) Find the terminal voltage of the motor.
(*d*) Find the mechanical power developed by the motor in watts.

17. What is the efficiency of the generator in Prob. 16 and what is the efficiency of the motor? Assume that all losses except the heat losses due to RI^2 are negligibly small.

Note: Since the power required to maintain the magnetic fields shown in Fig. 10 is not taken into account, the efficiencies as here determined are "armature" efficiencies.

18. The generator armature conductor of Fig. 10 is attached to a prime mover which drives the conductor forward (to the right) and backward (to the left) in simple harmonic motion such that the displacement or distance (from some $t = 0$ reference position) is

$$x = 2 \sin (5t) \quad \text{meters}$$

if t is expressed in seconds.

The length of the generator armature conductor is 1 m, and the generator flux density (uniformly distributed) is 0.20 weber/sq m.

(*a*) What is the value of the voltage generated in this conductor expressed as a function of time?

(*b*) Describe the general nature of the motion of the motor conductor under this condition of operation.

19. Assume that the elementary generator-motor combination shown in Fig. 10 is operating under conditions similar to those described in Prob. 18, the specific operating conditions being:

Generator Data	*Motor Data*
$R_{\text{gen.}} \doteq 0$ (negligibly small)	$R_m = 0.04$ ohm (only R present)
$B_{\text{gen.}} = 0.20$ weber/sq m	$B_m = 0.40$ weber/sq m
$l_{\text{gen.}} = 1.0$ m	$l_m = 2.0$ m
$v_{\text{gen.}} = 10 \cos (5t)$ m/sec	$v_m = 2 \cos (5t)$ m/sec

In the expressions for velocity, time t is to be expressed in seconds, thereby making the length of time of one complete cycle of motion $2\pi/5$ sec.

Find the time-averaged value of the mechanical power developed by the motor under the specified conditions of operation given above. *Note:* By time-averaged value in a case of this kind is meant the average value over any complete number of working cycles, that is, over any integral number of $2\pi/5$-sec periods of time in this case.

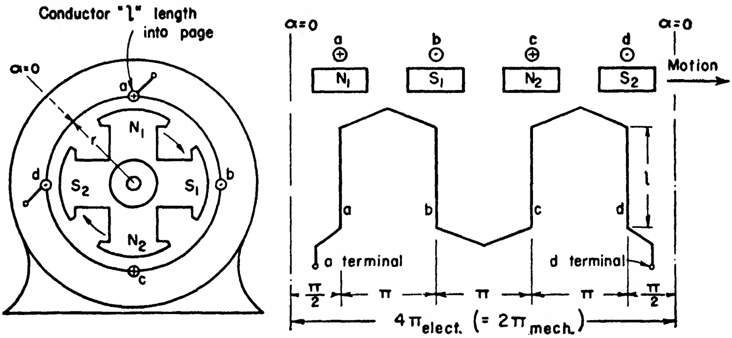


FIG. 11. A four-pole, four-conductor a-c generator of the revolving-field type. See Prob. 20.

20. Consider the 4-pole a-c generator shown in Fig. 11 in which the space distribution of the magnetic flux density produced by the rotating field structure is

$$B = B_{\text{max.}} \sin \alpha$$

where α is reckoned as zero midway between any two poles of the rotating field structure.

The stationary armature winding in which voltage is generated consists of four active conductors which are spaced one pole span or π electrical radians apart, as shown in the developed diagram of the armature winding in Fig. 11. In order to be able to represent the space distribution of the magnetic flux density as simply $B = B_{\text{max.}} \sin \alpha$, it is plain that each pair of poles of the machine must be considered as 2π electrical radians even though these 2π radians occupy only π mechanical radians of the machine. The known data are:

$B_{\text{max.}}$ (in $B = B_{\text{max.}} \sin \alpha$) is 1.0 weber/sq m
effective axial length of each active conductor, $l = 25$ cm
speed of rotation $= 1800$ rpm
radius at which conductors are from axis of rotation, $r = 20$ cm

(*a*) What is the maximum instantaneous voltage which appears at the terminals of the armature winding?

(*b*) Write an expression for e_{ad} as a function of time, choosing any arbitrary $t = 0$ reference time. (The angular velocity of this time variation must be expressed in electrical radians per second.)

21. In Fig. 12, the conducting loop *ABCD* originally links with the flux of the horseshoe magnet, which is shown in cross-section by the N and S poles. The loop *ABCD* is moved to the right as indicated to a position where no flux is linked by this loop. During the movement to the right, the loop *ABCD* remains closed through the conductor *E* over which the clip end of loop *ABCD* slides.

Is a voltage generated in loop *ABCD* during the movement of the loop described above? Explain.

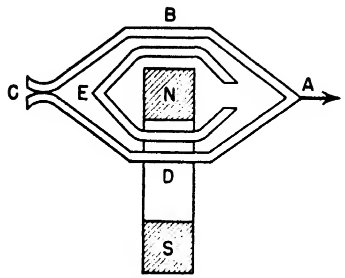


FIG. 12. See Prob. 21.

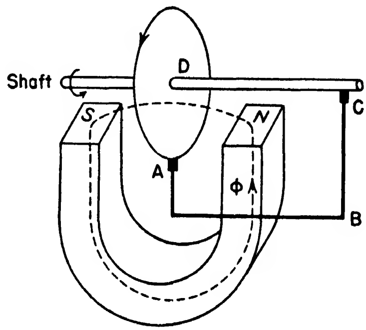


FIG. 13. See Prob. 22.

22. In Fig. 13, a metallic disk *D* is so located relative to the magnetic field established by the horseshoe magnet (N-S) that a portion of the flux passes through the metallic disk.

(*a*) Will circulating current be established in the *ABCD* loop as a result of the rotation of the disk indicated in Fig. 13?

(*b*) If a circulating current is established in the *ABCD* loop, in what direction around the *ABCD* loop is the current directed, clockwise or counterclockwise?

(*c*) Can the phenomenon be explained on the basis of change of flux through the conducting loop?

CHAPTER XI

Ferromagnetic Circuits

1. Ferromagnetism. The relative permeability μ_r, which originally entered into the definition of **B** ($\mathbf{B} = \mu_0\mu_r\mathbf{H}$), has thus far been considered to be unity—the value which is applicable to the vast majority of material media. Of the chemical elements, only iron, nickel, and cobalt exhibit relative permeabilities which are essentially different from unity. These three elements (and a few specially prepared *alloys* of other metals) may under certain operating conditions exhibit values of μ_r that exceed 100,000. In all cases where μ_r is essentially different from unity, special methods of analysis must be employed because, where μ_r is significantly different from unity, it is also far from being a constant. Materials like iron having relative permeabilities greater than unity are called ferromagnetic materials.

In the unmagnetized state, ferromagnetic materials may be thought of as consisting of billions of tiny current loops per cubic centimeter which are oriented at random as pictured in Fig. 1. In a totally random state, these current loops which are due to spinning electrons do not make themselves felt outside the material because the mmf's produced by the loops are canceled so far as the outside region is concerned by the hit-and-miss orientations of the billions of loops. Any path that might be selected through the material would encounter as many + to − *magnetic batteries* as it would − to + magnetic batteries; therefore, by the circuital law of magnetism, the resultant **H** vector around any closed loop which passed through the unmagnetized material would experience a *net* magnetic potential rise of zero.

The molecular structure of most materials is such that the mmf's produced by the current loops within the molecule are completely canceled owing to the balanced manner in which these tiny current loops are aligned within the molecular or atomic structure; or, as the physicist might say, the electron spins are balanced in most materials. But in iron, nickel, and cobalt these tiny current loops are not balanced, and as a result molecules of these materials can be aligned by the

magnetic forces which act on the current loops when these loops are in the presence of a magnetic field.

The tiny current loops in ferromagnetic materials experience magnetic forces which tend to so align them that the faces of the loops are normal to the magnetic lines $\mathbf{H}_0$ of the external field as shown in Fig. 2. Since these current loops are themselves sources of magnetic flux, ferromagnetic materials introduce additional mmf into the $\mathbf{H}$ paths which pass through these materials and thereby increase the magnetic flux density over the value which would be present in this same region had

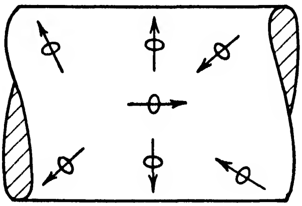


FIG. 1. Random-directed subatomic current loops in the unmagnetized state.

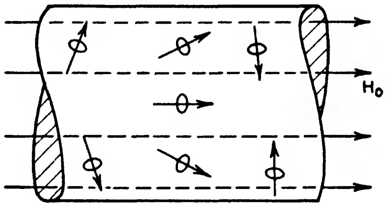


FIG. 2. Current loops partially aligned under the influence of $\mathbf{H}_0$.

the ferromagnetic material not been present. The manner in which this *increase in flux density* is accounted for is the subject of the following section.

2. Relative Permeability μ_r. In free space (or in most materials) the magnetic flux density (in mks units) is

$$\mathbf{B}_0 = \mu_0 \mathbf{H}_0 \quad \text{free-space webers/sq m} \tag{1}$$

where μ_0 is the permeability of free space ($4\pi \times 10^{-7}$ in rationalized mks units)

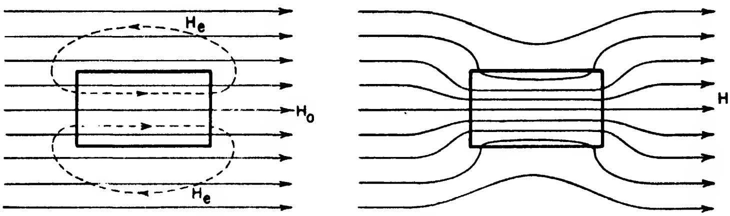
$\mathbf{H}_0$ is the free-space value of the magnetic field intensity expressed in amperes per meter and which has previously been symbolized as $\mathbf{H}$.

By assigning to ferromagnetic materials a *relative* permeability which is greater than unity, the increase in B due to the internal mmf's possessed by these materials may be taken into account, without the necessity of performing the impossible task of counting the billions of tiny *magnetic batteries* which are inherently present in ferromagnetic materials. The procedure is simply to write

$$\mathbf{B} = \mu_0 \mu_r \mathbf{H}_0 \quad \text{or} \quad \mu_r = \frac{\mathbf{B}}{\mu_0 \mathbf{H}_0} = \frac{B}{B_0} \tag{2}$$

where B is the magnitude of the resultant flux density in the ferromagnetic material and B_0 is the free-space magnitude of the flux density which would result *if the material were replaced by free space and subjected to the same value of* $\mathbf{H}_0$.

Although this method of accounting for the *increase* in flux density is algebraically simple, it has certain practical limitations. An example of the complications which may arise from the introduction of a piece of iron into an otherwise uniformly distributed magnetic field is shown in Fig. 3. The internal current loops of the iron molecules (or crystals) become oriented under the influence of H_0 and in so doing become a seat or source of magnetic lines H_e which disrupt the original flux distribution in the region nearby as shown in Fig. 3-b. It will be observed that the



(a) SHOWING BOTH H_0 AND H_e (b) SHOWING THE RESULTANT OF $H_0 + H_e$

FIG. 3. Illustrating the effect of introducing a short piece of ferromagnetic material into an otherwise uniformly distributed magnetic field.

flux density (as represented by the space density of the lines drawn in Fig. 3-b) is about twice as great in the iron as it was in this space before the iron was introduced, indicating that the relative permeability of the iron is about 2. This same piece of iron might, if operated under different conditions exhibit, μ_r's higher than 10^5, the reason being that the arrangement shown in Fig. 3 is not well adapted to obtaining a high ratio of B/B_0.

Since magnetic flux lines must be continuous, it is physically impossible to have a large resultant flux density B in the piece of iron shown in Fig. 3 without at the same time having a high value of B_0. Physical arrangements which are better adapted to obtaining high ratios of B/B_0 (and hence high values of μ_r) are given in the following sections of this chapter.

Example. Let it be required to find the relative permeability of the iron sheath under the operating conditions shown in Fig. 4 if it is known that the flux threading through the iron sheath per meter axial length of sheath is 5.6×10^4 maxwells.

The space-averaged value of the actual or resultant flux density in the iron sheath is

$$B = \frac{\phi}{A} = \frac{(5.6 \times 10^4)(10^{-8})}{(1.0)(0.002)} = 0.28 \quad \text{weber/sq m}$$

An application of the circuital law of magnetism to the symmetrical arrangement shown in Fig. 4 will show that

$$H_0 = \frac{I}{2\pi r} = \frac{30}{0.06\pi} = \frac{500}{\pi} \quad \text{amp/m} \quad \text{(directed as shown)}$$

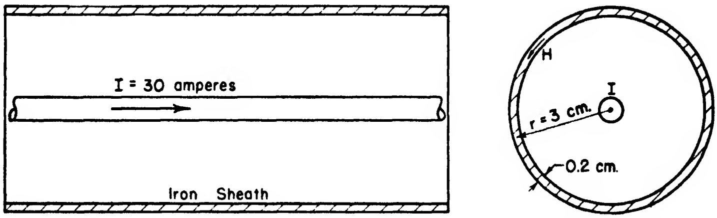


FIG. 4. A long straight current-carrying conductor surrounded by an iron sheath.

at the center of the iron sheath. For the narrow sheath employed in this case, the above value of H_0 represents a fairly good space-averaged value of H_0 throughout the entire region occupied by the sheath.

$$B_0 = \mu_0 H_0 = (4\pi \times 10^{-7})\left(\frac{500}{\pi}\right) = 2 \times 10^{-4} \quad \text{weber/sq m}$$

$$\mu_r = \frac{B}{B_0} = \frac{0.28}{2 \times 10^{-4}} = 1400 \quad \text{(a numeric)}$$

These equations mean that the resultant value of B (and hence of ϕ) in the iron sheath is 1400 times as great as the value of B_0 (and hence ϕ_0) which would have existed in this same region had the iron sheath not been present.

In many practical cases the important problem is the reverse of that presented in the above example, namely, given data from which μ_r and H_0 may be evaluated, find the resultant flux density in magnetic materials and the magnetic flux which pierces some particular cross-sectional area of this material. In most practical problems, the data come to us in such a way that the actual evaluation of μ_r is unnecessary, as will be shown presently.

3. Incremental Permeability. The relative permeability defined in equation (2) is sometimes referred to as the d-c permeability to distinguish it from *incremental* (relative) *permeability* which is symbolized and defined by

$$\Delta\mu_r = \frac{\Delta B}{\Delta B_0} = \frac{\Delta B}{\mu_0\, \Delta H_0} \tag{3}$$

As implied by the Δ's, incremental permeability is a measure of the *change* in B, (ΔB), relative to the *change* in $\mu_0 H_0$, $(\mu_0\, \Delta H_0)$, which produces the change in B.

The importance and physical significance of *incremental permeability* will be established more clearly in the following chapter, where it is shown to be a significant factor in determining the self-inductance of iron-cored coils which are energized with a current composed of both a d-c component (which establishes a μ_r) and an a-c component (which establishes a $\Delta\mu_r$).

4. Behavior of Magnetic Flux in and near Iron Cores. In order to have an intelligent understanding of the approximations which are made in reducing iron-core magnetic *field* problems to simple *circuit* problems,

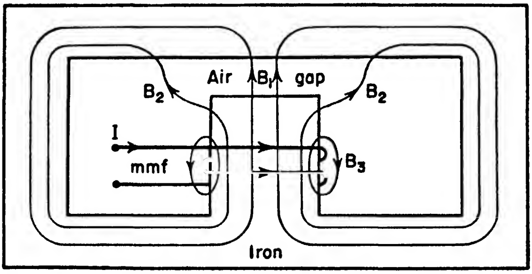


FIG. 5. Flux paths in an iron core with air gap.

some knowledge of the behavior of magnetic flux in the vicinity of iron cores is essential. We can readily visualize the magnetic paths which cross the air gap in Fig. 5 along the B_1 paths as crossing the boundary between the iron pole and the air in straight parallel lines. In this center-of-pole region the associated **H** vectors undergo a decided change in magnitude but no change in direction since they cross the boundary at right angles.

From the fact that the normal component of **B**, (B_n), is continuous as it crosses a boundary from a region of μ_{r1} to a region of μ_{r2}, it follows that the normal components of **H**, (H_n), must suffer an abrupt change in magnitude which is inversely proportional to the μ_r's on either side of the boundary since $H_n = \mu_0\mu_r B_n$. In Fig. 5, for example, H_n may increase by a factor of 100 or more as it crosses the boundary from the top of the iron pole into the air gap. Since $B_n = \mu_0\mu_{r1}H_{n1} = \mu_0\mu_{r2}H_{n2}$,

$$\frac{H_{n2}}{H_{n1}} = \frac{\mu_{r1}}{\mu_{r2}} \tag{4}$$

Some of the total flux linking with the mmf coil of Fig. 5 is shown as *fringing* out at the air gap along the B_2 paths. The **H** vectors which are

associated with the B_2 paths have both normal and tangential components to the boundary between the iron core and the air. The **H** vectors in this case suffer both change in magnitude and change of direction as they cross the boundary.

The change in magnitude and direction of the **H** vectors as they cross from a μ_{r1} region to a μ_{r2} region is shown in Fig. 6, where it is assumed that the tangential components ($H_1 \cos \theta_1$) and ($H_2 \cos \theta_2$) are equal; since the $H\, \Delta l$ magnetic potential drop between two points *on* the boundary surface is either $H_2 \cos \theta_2\, \Delta l$ or $H_1 \cos \theta_1\, \Delta l$ as indicated in

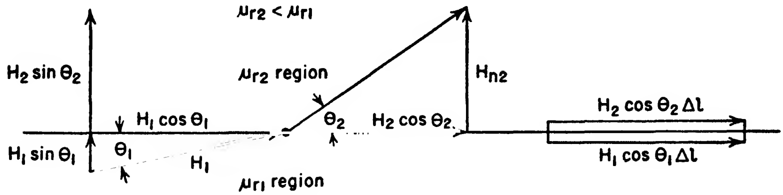


FIG. 6. Illustrating the change in magnitude and direction of an **H** vector as it crosses from a μ_{r1} region to a μ_{r2} region.

Fig. 6. An interpretation of Fig. 6 will show that on the basis of $H_1 \cos \theta_1 = H_2 \cos \theta_2$

$$\frac{\tan \theta_1}{\tan \theta_2} = \frac{H_1 \sin \theta_1}{H_2 \sin \theta_2} = \frac{H_{n1}}{H_{n2}} = \frac{\mu_{r2}}{\mu_{r1}} \tag{5}$$

or

$$\tan \theta_2 = \frac{\mu_{r1}}{\mu_{r2}} \tan \theta_1 \tag{6}$$

Thus if μ_{r1} is relatively large and the μ_{r2} region is air, a flux line (either **H** or **B**) may approach the boundary surface from inside the μ_{r1} region at a very small angle (θ_1) and yet leave this boundary surface at an angle which approaches 90°.

The actual distribution of the magnetic flux in an air-gap region between two ferromagnetic surfaces is difficult to determine but it can be estimated reasonably well by means of magnetic field maps. Magnetic field maps show magnetic flux lines crossing planes of equal magnetic potential at right angles in the same manner as electric field maps show electric flux lines crossing planes of equal electric potential at right angles.

It is by means of field maps that designers have worked out approximations (which we shall presently apply) to account for the fringing of magnetic flux at air gaps. A map of this kind is shown in Fig. 7 for the air-gap region between the pole and the armature teeth of a rotating

electrical machine. Since the method of constructing a map of this kind is identical with the curvilinear-square method previously considered, the details of the method will not be repeated here. It should be noted, however, that the flux lines in Fig. 7 either enter or leave the iron surfaces (shown in heavy lines) at essentially 90° in all places along the iron surfaces.

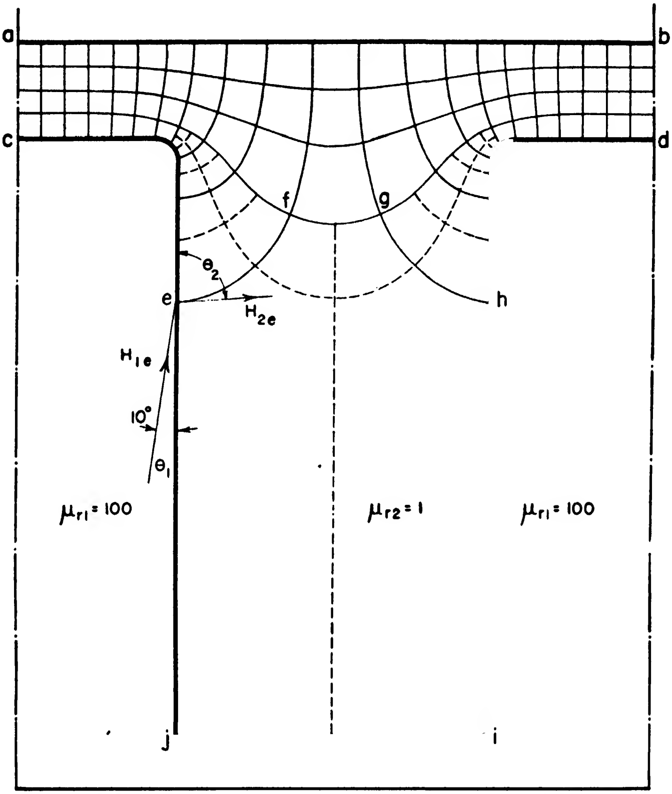


FIG. 7. Curvilinear-square map of the magnetic field in the air gap between the pole face (*ab*) and the armature teeth (*cejihd*) of a rotating machine.

Use is made of equation (6) at a point like *e* along the side of the tooth where $\mathbf{H}_{1e}$ approaches the tooth surface at an angle of 10° and enters the air gap as $\mathbf{H}_{2e}$ at an angle $\theta_2 = 87°$ as determined by equation (6).

5. Elementary Magnetic Circuit Concepts. In magnetic field theory each path traced out by an exploring compass is actually a magnetic *circuit*, and in this sense there is an infinite number of parallel magnetic circuits that link with the current loops NI which establish the magnetic field. Obviously, this infinite number cannot be handled quantitatively, so we think of 1 maxwell or 1 weber of magnetic flux as a tube

of finite dimensions which closes on itself and which contains all the magnetic paths that could possibly be traced out, without crossing the boundary surface of this closed tube. The closed tube may have any cross-sectional area A_t which we choose, and our choice depends on the quantity of magnetic flux $\Delta\phi$ which we wish to have contained within the closed boundary surface of the tube. However, $\Delta\phi$ having been selected, the tube will be of such dimensions that the cross-sectional area A_t at all points along the length of the tube will equal $\Delta\phi/B_t$, where B_t is the space-averaged flux density over the cross-sectional area A_t. Since magnetic flux lines from the outside can never enter the tube

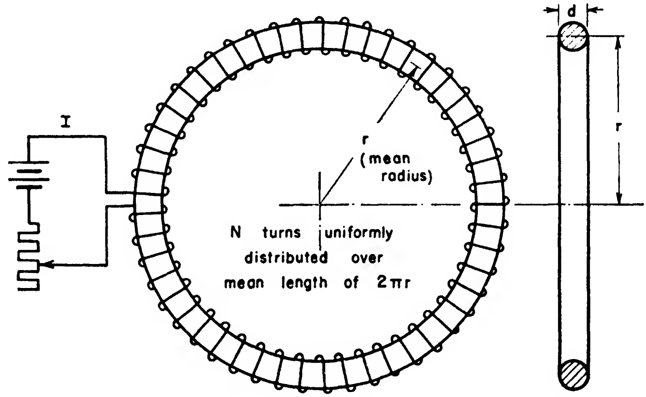


Fig. 8. An elementary magnetic circuit.

(because it is bounded by flux lines and these lines cannot cross), A_t grows larger in regions where the field *spreads out* and B_t grows smaller, and vice versa.

A good approach to a tube of magnetic flux of constant cross-sectional area is the region which is surrounded by the distributed coil shown in Fig. 8. In this case it is plain that the magnetic field intensity H_0 is directed along circular paths which range in length from $2\pi(r - 0.5d)$ to $2\pi(r + 0.5d)$ and that, if $d < < r$, the value of H_0 along the $2\pi r$ path is a good space-averaged value of the magnetic field intensity throughout the region bounded by the turns of wire. Applying the circuital law of magnetism to the $2\pi r$ circular path in Fig. 8, we find that the mmf linked with this path is NI amp-turns. Hence

$$H_0 = \frac{NI}{2\pi r} \quad \text{amp-turns/m} \tag{7}$$

and

$$B_0 = \mu_0 H_0 = \mu_0 \frac{NI}{l} \tag{8}$$

where l is the length of the magnetic circuit, namely, $2\pi r$ m.

$$B = \mu_0\mu_r \frac{NI}{l} = \mu \frac{NI}{l} \tag{9}$$

where $\mu_0\mu_r$ is symbolized as μ simply for convenience in writing.

The amount of magnetic flux crossing any of the $\pi d^2/4$ cross-sectional areas of the magnetic circuit shown in Fig. 8 is

$$\phi = BA = \mu \frac{NI}{l} A \tag{10}$$

which may be rearranged to read as follows

$$\phi = \frac{NI}{\dfrac{l}{\mu A}} = \frac{\text{mmf}}{\mathcal{R}} \tag{11}$$

$\mathcal{R} = l/\mu A$ is called the reluctance of the magnetic circuit. The analogy between $\mathcal{R} = l/\mu A$ and resistance $R = l/\gamma A = \rho(l/A)$ is clear, and, even though equation (11) is sometimes referred to as Ohm's law of magnetic circuits, the relationship is rather useless from a practical point of view. In order to find $\mathcal{R}$, μ_r must be evaluated, and in most practical magnetic circuit problems the solutions are obtained before μ_r can be evaluated.

Example. Let it be required to determine the flux which crosses any $\pi d^2/4$ area in the region bounded by the current-carrying coil in Fig. 8 under the following conditions.

$$N = 1200 \quad I = 100 \text{ ma} \quad d = 0.04 \text{ m} \quad r = 0.20 \text{ m}$$

The relative permeability μ_r for this particular iron core energized with 120 amp-turns is 8330.

The last bit of information is usually lacking in practical problems, and it causes this method of evaluating magnetic flux to be of little consequence.

The magnetomotive force acting on the $2\pi r$ circular path is

$$\text{mmf} = NI = 1200 \times 0.10 = 120 \quad \text{amp-turns}$$

which in this case is analogous to a long battery of 1200 cells (each of 0.1 amp-turn potential) short-circuited on itself.

The reluctance of the magnetic path is

$$\mathcal{R} = \frac{l}{\mu A} = \frac{2\pi r}{\mu_0\mu_r\left(\dfrac{\pi d^2}{4}\right)} = \frac{2\pi(0.20)}{(4\pi \times 10^{-7})(8330)(0.0004\pi)}$$

$$\mathcal{R} = 95{,}700 \quad \text{(rationalized mks units of reluctance)}$$

No name has been given to either the rationalized or unrationalized mks unit of reluctance but since it is dimensionally mmf/flux we may call the rationalized mks unit of reluctance amp-turns/weber. In this particular case the reluctance of the circuit corresponds to the internal resistance of the short-circuited battery, and in order to find the magnetic flux which is analogous to the current that would flow through the short-circuited battery we employ equation (11):

$$\phi = \frac{\text{mmf}}{\mathcal{R}} = \frac{120}{95{,}700} = 1.257 \times 10^{-3} \quad \text{weber} \quad (\text{or } 125{,}700 \text{ maxwells})$$

6. *B–H* Curves. Since the *increase* in flux density which is produced by the alignment of the sub-atomic current loops in ferromagnetic

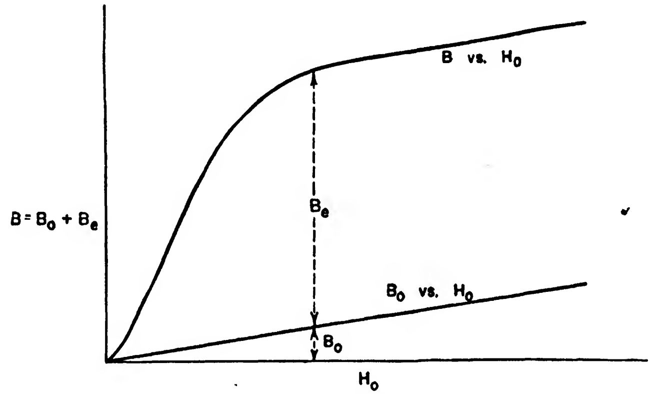


FIG. 9. General shape of B–H curve showing the contribution (to the resultant B) of the ferromagnetic material.

materials cannot be determined theoretically, recourse must be had to experimental data which are usually presented in the form of a curve of resultant flux density B, for a particular material, plotted versus the magnitude of H which is employed to produce B. These curves are known as B–H curves, the general shape of which is shown in Fig. 9.

Although B_0 $(= \mu_0 H_0)$ versus H_0 is a straight line, the resultant B in ferromagnetic materials $(B = B_0 + B_e)$ versus H_0 is far from being a straight line. The lower *knee* of Fig. 9 (near $H_0 = 0$) indicates that the orientation of the sub-atomic current loops is not so pronounced at very small values of H_0 as it is for slightly greater values of H_0, where an increase of H_0 is accompanied by a rapid and almost linear increase in the resultant flux density B. This mid-portion of the B–H curve, where B increases rapidly for small changes in H_0, is the portion normally used in practice; it is referred to as the straight-line portion of the B–H curve even though it is usually not so straight as the portion of the curve

beyond the upper *bend* of the curve. The high values of H_0 and the accompanying high values of exciting current usually make it unprofitable to use the material above the upper bend or knee of the curve.

The upper *bend* between the two straight-line portions of the B–H curve indicates that a saturation effect is present beginning near the lower end of the upper bend. The indication is that the ferromagnetic material is failing to provide any *further increase* in the resultant value of the flux density B. In other words, all the sub-atomic current loops which are going to line up readily under the influence of the magnetizing force H_0 have done so at the H_0 value which gives a maximum B_e. It will be realized that, with the billions of tiny current loops which are involved, the saturation point is not clearly defined, but for all practical purposes the ferromagnetic material has contributed all the increase

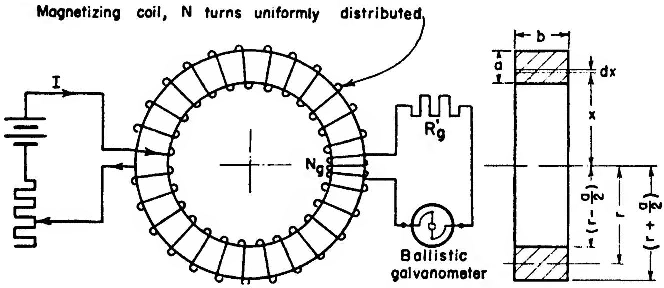


FIG. 10. Experimental method for determining B-H curves.

in B that it is capable of at the upper end of the upper bend of the B–H curve. The point where saturation is essentially complete is shown in Fig. 9, where the B_e ordinate is labeled.

7. Experimental Determination of B–H Curves. Several experimental methods are available for obtaining the B versus H relationship of ferromagnetic materials. One of these methods is indicated in Fig. 10. Although the method shown in Fig. 10 is not well adapted to high-speed production of B–H curves, it is one of the more accurate methods and, since it is inherently a very simple method, the details will be considered briefly.

The magnetic circuit shown in Fig. 10 is essentially the same as the magnetic circuit shown in Fig. 8 with a second coil of N_g turns encircling the flux established by the distributed primary winding of N turns. The two coils are not connected electrically in any way.

Since a ballistic galvanometer reads in direct proportion to the amount of charge ($Q_g = I_g \Delta t$) which passes through it, a ballistic galvanometer may be used to measure *change of magnetic flux* in the following manner.

When the primary current (I in Fig. 10) is either established or reduced to zero in a short period of time, say Δt,

$$\text{deflection of the galvanometer, } D = K_1 Q_g = K_1 I_g \, \Delta t \qquad (12)$$

where K_1 depends solely upon the construction of the galvanometer.

The voltage which is generated in the N_g coil by the change of flux which accompanies any change in the exciting current I is

$$E_g = N_g \frac{\Delta\phi}{\Delta t} \quad \text{(time-averaged value)} \qquad (13)$$

The time-averaged value of current which flows in the galvanometer circuit having R_g ohms of resistance is

$$I_g = \frac{E_g}{R_g} = \frac{N_g \, \Delta\phi}{R_g \, \Delta t} \qquad (14)$$

By combining equations (12) and (14) we have

$$D = K_1 \frac{N_g \, \Delta\phi}{R_g \, \Delta t} \Delta t = \frac{N_g \, \Delta\phi}{K} \qquad (15)$$

where $K = R_g/K_1$ is the galvanometer sensitivity which can readily be determined as shown in the following example if it is not already known. Knowing K, we know the *change-of-flux linkage* in the galvanometer circuit *per scale division* of the galvanometer reading.

The magnetizing force H_0 in the sample which is under investigation is simply

$$H_0 = \frac{NI}{2\pi r} \quad \text{amp-turns/unit length} \qquad (16)$$

and if $r >> a$, as is assumed in equation (16), the space-averaged value of the magnetic flux density in the sample is

$$B = \frac{\Delta\phi}{A} = \frac{KD}{N_g A} \qquad (17)$$

where $A\,(= ab$ in Fig. 10) is the cross-sectional area of the sample.

Thus H_0 (or simply H) can be measured in terms of N, I, and r; and B in terms of K, D, N_g, and A. A series of readings may be taken and the results plotted in the form of a B–H curve, and it is in the form of B–H curves that the experimental results are usually presented. (See Fig. 11.) Some of the refinements which are at times employed in space-averaging H and B when r is not large as compared with a in Fig. 10 are considered in Prob. 13 at the close of the chapter.

Example. Let it be required to find the sensitivity of a galvanometer arranged as shown in Fig. 10 from the following data:

Non-ferromagnetic core dimensions: $r = 20$ cm, $a = 4$ cm, $b = 5$ cm
Number of turns of magnetizing winding, $N = 1000$
Number of turns of galvanometer test coil, $N_g = 500$

An incremental *change* of 1 amp of magnetizing current I is observed to be accompanied by a galvanometer deflection D of 10 scale divisions.

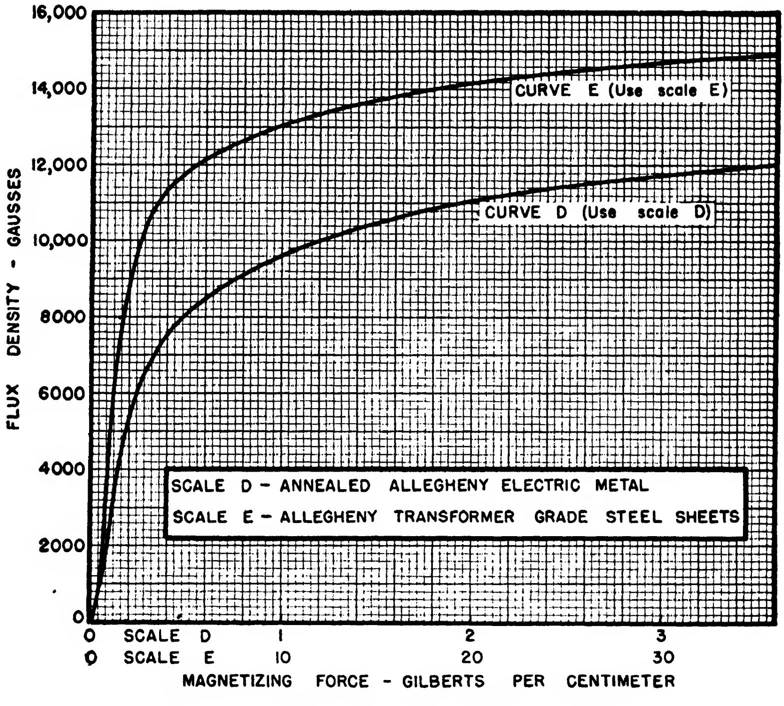


FIG. 11. *B–H* curves of two different metals.

The change of magnetic flux which in turn accompanies a galvanometer deflection of 10 scale divisions may be calculated straightforwardly from the known data. For an incremental change of 1 amp of exciting current

$$\Delta H_0 = \frac{N\Delta I}{2\pi r} = \frac{(1000)(1)}{2\pi(0.2)} = \frac{2500}{\pi} \quad \text{amp-turns/m}$$

along the $2\pi r$ path of Fig. 10. The associated change in flux density is

$$\Delta B = \mu_0 \Delta H_0 = 4\pi \times 10^{-7} \frac{2500}{\pi} = 10^{-3} \quad \text{webers/sq m}$$

or 10 maxwells/sq cm since galvanometer sensitivities are usually reckoned in maxwell-turns per scale division.

$$\Delta\phi = (\Delta B)A = 10(4 \times 5) = 200 \quad \text{maxwells}$$

Hence the galvanometer sensitivity follows directly from equation (15) as

$$K = \frac{N_g \Delta\phi}{D} = \frac{500 \times 200}{10} = 10{,}000 \quad \text{maxwell-turns/scale division}$$

8. Secondary Units of B and H that Are Used in Practice. B–H curves are very seldom given in webers per square meter plotted against ampere-turns per meter because various secondary units of B and H are well

TABLE I

MAGNETIC FLUX DENSITY B

Multiply B by number of → / to obtain B in ↓	gauss or maxwells/sq cm or lines/sq cm	maxwells/sq in or lines/sq in	webers/sq m
gauss or lines/sq cm	1	0.1550	10^4
lines/sq in	6.452	1	6.452×10^4
webers/sq m	10^{-4}	0.1550×10^{-4}	1

established and are used by practicing engineers. The solution of practical magnetic circuit problems requires a knowledge of these secondary units to the extent at least of being able to visualize the relative sizes of these

TABLE II

MAGNETIZING FORCE H OR MAGNETIC POTENTIAL GRADIENT

Multiply H by number of → / to obtain H in ↓	gilberts/cm or oersteds	amp-turns/in	amp-turns/m
gilberts/cm	1	0.4950	1.257×10^{-2}
amp-turns/in	2.021	1	2.540×10^{-2}
amp-turns/m	79.58	39.37	1

units. The secondary units of B and H most commonly used are shown in Tables I and II together with the primary units of B and H in the rationalized mks system of units which has been employed almost exclusively in this text.

Some use has been made of the fact that the maxwell (or line) of magnetic flux as a unit of measure is 10^{-8} times smaller than the weber. When the term *line* is used quantitatively it is understood that a *tube* of flux containing 1 maxwell is actually meant. The maxwell per square centimeter is the primary unit of magnetic flux density in the ab-cgs system of units and is often called a *gauss*, as shown in Table I. Any of the conversion factors shown in Table I can readily be substantiated if it is recognized that webers/line = 10^{-8}, sq m/sq cm = 10^{-4}, and sq in./sq m = 0.155×10^4. For example, in the conversion from lines per square inch to webers per square meter one merely writes

$$(\text{No. of})\,\frac{\text{webers}}{\text{sq m}} = \frac{\cancel{\text{lines}}}{\cancel{\text{sq in.}}} \times \frac{\text{webers}}{\cancel{\text{line}}} \times \frac{\cancel{\text{sq in.}}}{\text{sq m}}$$

$$= \frac{\text{lines}}{\text{sq in.}} \times 10^{-8} \times (0.155 \times 10^4)$$

Then, as Table I indicates, to obtain $B_{(\text{webers/sq m})}$ multiply $B_{(\text{lines/sq in.})}$ by the factor 0.1550×10^{-4}. The other conversions shown in Table I follow in a similar manner.[1]

The gilbert is the primary unit of mmf in the unrationalized ab-cgs system of units, and as a unit of measure it is 1.257 (or 0.4π) times *smaller* than the ampere-turn which has been used throughout this text and which is a much more logical unit from a practical point of view. If it is recognized that

$$(\text{No. of})\text{ gilberts} = 1.257\, NI_{\text{amp}} \quad \text{or} \quad \frac{\text{gilberts}}{\text{amp-turn}} = 1.257 \tag{18}$$

any of the conversions shown in Table II may be evaluated. For example, to convert H expressed in gilberts per centimeter to H in ampere-turns per meter we write

$$(\text{No. of})\,\frac{\text{amp-turns}}{\text{m}} = \frac{\cancel{\text{gilberts}}}{\cancel{\text{cm}}} \times \frac{\text{amp-turns}}{\cancel{\text{gilbert}}} \times \frac{\cancel{\text{cm}}}{\text{m}}$$

$$= \frac{\text{gilberts}}{\text{cm}} \times \frac{1}{1.257} \times 100$$

Then, as Table II indicates, to obtain $H_{(\text{amp-turns/m})}$ multiply $H_{(\text{gilberts/cm})}$ by the factor 100/1.257 or 79.58.

Example. Let it be required to evaluate the relative permeability of the two grades of metals for which the B–H curves shown in Fig. 11 apply; the

[1] The general method of presentation shown in Tables I and II is adapted from Eshbach's *Handbook of Engineering Fundamentals*, John Wiley and Sons.

evaluation to be performed at B = 10,000 gauss = 10,000 maxwells/sq cm = 1 weber/sq m.

In evaluating $\mu_r = B/B_0 = B/\mu_0 H_0$, *the B's must be expressed in the same units.*

Since μ_0 is unity in the unrationalized ab-cgs system of units and since gauss and gilberts per centimeter are primary units of B and H respectively in this system of units, the curves are well adapted to this system of units. Thus

$$\mu_{r1} \text{ (of Allegheny electric metal)} = \frac{B}{\mu_0 H_0} = \frac{B}{H_0} = \frac{10{,}000}{1.2} = 8330$$

$$\mu_{r2} \text{ (of transformer grade steel)} = \frac{B}{\mu_0 H_0} = \frac{B}{H_0} = \frac{10{,}000}{2.6} = 3850$$

As an example of conversion of units, we shall make these same evaluations in rationalized mks units where B = 10,000 gauss = 1 weber/sq m. In this case, we first find $H_0 = H$ in ampere-turns per meter, as

$$H_{(\text{amp-turns/m})1} = H_{(\text{gilberts/cm})1} \times 79.58 = 1.2 \times 79.58 = 95.5 \quad \text{amp-turns/m}$$

$$H_{(\text{amp-turns/m})2} = H_{(\text{gilberts/cm})2} \times 79.58 = 2.6 \times 79.58 = 207 \quad \text{amp-turns/m}$$

Then

$$\mu_{r1} = \frac{B}{\mu_0 H_0} = \frac{1}{(4\pi \times 10^{-7})(95.5)} = 8330$$

$$\mu_{r2} = \frac{B}{\mu_0 H_0} = \frac{1}{(4\pi \times 10^{-7})(207)} = 3850$$

The symbol H_0 which has been used thus far in this chapter to emphasize the fact that it does *not* include the mmf rise produced by the sub-atomic current loops in the ferromagnetic material is generally referred to simply as H, the understanding being that $H = H_0$ is the free-space value of H produced only by the exciting coil (or coils) which energize the magnetic circuit.

9. Simple Magnetic Circuit Homogeneous in the l Direction. Many magnetic cores employed in transformers and other magnetic devices are rectangular in shape as indicated in Fig. 12 and are composed of thin laminations of sheet steel, often of No. 29 gage material which is 0.014 in. in thickness. In reducing the arrangement shown in Fig. 12 to a simple *circuit,* two assumptions are made:

(1) The leakage flux ϕ_{11} which links with the exciting winding but which does *not* follow the circuit path of length l is of no significance in determining the useful flux ϕ_M.

(2) The insulation on the surface of the stacked laminations (which is often only a naturally formed oxide) reduces the stacked core to an

equivalent solid core by a stacking factor (S.F.) which has a value which ranges between 0.85 and 0.95 depending on the thickness of the laminations and the amount of insulation present on the surface of each lamination.

In (1) we are saying that the $H\ \Delta l$ magnetic potential drop caused by the leakage flux ϕ_{11} is negligible in so far as it affects the Hl drop due to ϕ_M around the entire length of the core. The small effect of a slightly higher value of H in the section of the core through which both ϕ_{11} and ϕ_M exist tends to make ϕ_M actually less, by about 1 or 2 per cent, than our calculated value will be if the calculated value is determined on the basis of a simple series circuit.

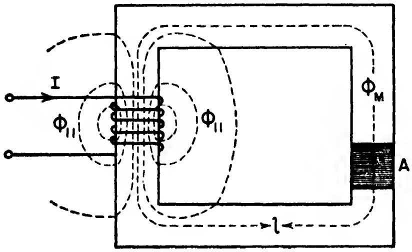


FIG. 12. Simple magnetic circuit, homogeneous along the l dimension.

In (2) we neglect the flux which arrives at the leg opposite the exciting coil by way of the nonferromagnetic insulation which is, of course, in parallel with the solid iron paths. This flux would in most cases be less than ½ per cent of the ϕ_M which exists in the solid iron, and whatever its value it tends to compensate for the approximation made in (1).

To the accuracy which can be expected where B–H curves are used, the above assumptions are wholly justified, and as a result of making these assumptions we have reduced a very difficult problem to a very simple one. In order to find ϕ_M for a specified value of NI (of the exciting coil) three simple steps are required:

(1) Find $H = NI/l$.

(2) With due regard for the units of H, look up B (on the B–H curve) which corresponds to this particular value of H.

(3) Find $\phi_M = BA_{\text{net}} = BA_{\text{gross}}(\text{S.F.})$, where S.F. means stacking factor.

In order to find NI which is required to establish a specified value of ϕ_M, the above procedure is reversed as shown below:

(1) Find $B = \phi_M/A_{\text{net}} = \phi_M/A_{\text{gross}}(\text{S.F.})$.

(2) With due regard for units, find the corresponding value of H from the B–H curve which is applicable to the specified material.

(3) Find NI as $[H_{(\text{amp-turns/u.l.})} \times l]$ where u.l. refers to the unit lengths used in expressing H.

Example. Let it be required to find the ampere-turns of the mmf coil in Fig. 12 which are needed to establish 822 kilolines of useful flux ϕ_M under the following specified conditions:

$$l = 30 \text{ in.} \quad A_{\text{gross}} = 3 \times 4 = 12 \text{ sq in.} \quad A_{\text{net}} = 12 \times 0.9 = 10.8 \text{ sq in.}$$

The core is stacked with No. 29 gage Allegheny transformer grade steel sheets which stack with a stacking factor of 0.9 in this case.

$$(1) \qquad B = \frac{822{,}000}{10.8 \times 6.45} = 11{,}800 \quad \text{lines/sq cm} \quad \text{(or gauss)}$$

(2) From curve E, Fig. 11,

$$H = 4.9 \text{ gilberts/cm} \quad (\text{for } B = 11{,}800 \text{ gauss})$$

From Table II,

$$H_{(\text{at./in.})} = H_{(\text{gilberts/cm})} \times 2.02 = 4.9 \times 2.02 = 9.9$$

$$(3) \qquad \text{Required } NI = H_{(\text{at./in.})}\, l_{\text{in.}} = 9.9 \times 30 = 297 \quad \text{amp-turns}$$

The result obtained is plainly not more accurate than the accuracy with which the B–H curve can be read; therefore a two-significant-figure result should probably be stated as 300 amp-turns.

A core 30 in. in length of the kind considered in the above example would not be truly homogeneous in the l direction if it were stacked with either two L-shaped laminations or with four straight pieces of laminations along the l direction. In either case the lamination insulation or the butt joints (or both) would introduce small air gaps into the flux paths in the l direction, and these air gaps even though very short in length might have a pronounced effect upon the number of ampere-turns required to establish a specified amount of magnetic flux in the core.

Transformer cores are, however, sometimes constructed by employing a continuous lamination (threaded around and around) through the coils, and in this type of core the magnetic circuit is homogeneous in the l direction, as has been assumed in this section.

10. Series Magnetic Circuits with Air Gaps. Many of the magnetic circuits employed in practice have air gaps in them across which the flux must pass. The magnetic circuits of rotating electrical machines, for example, must contain air gaps to permit rotation.

The introduction of an air gap into a magnetic circuit complicates the situation to some extent. In order to solve for the air-gap flux in Fig. 13 by simple circuit methods, the length of the air gap l_a must be relatively short compared to either dimension of the gross cross-sectional area of the ferromagnetic material which forms the core into which the air gap is introduced.

If l_a in Fig. 13 is less than about 10 per cent of the shorter of the two dimensions of A, the fringing of the flux which is inevitably present at the periphery of the gap may be accounted for quite accurately by increasing the gross cross-sectional area of the air gap (over that of the ferromagnetic material) as

$$\text{effective air-gap area } A_a = (a + l_a)(b + l_a) \tag{19}$$

where a and b are the dimensions of the gross cross-sectional area of the ferromagnetic core. (In Fig. 13, $A_{\text{gross}} = a \times b$.)

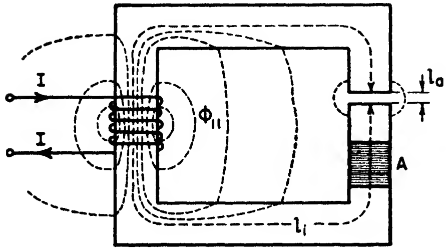


FIG. 13. Series magnetic circuit with air gap in the path of the flux.

After the approximation given in equation (19) has been made, the problem of finding the ampere-turns required to establish a specified value of air-gap flux ϕ_a is straightforward.

(1) Find H_il_i for the ferromagnetic material for the specified value of ϕ_a as shown in the preceding section; $B_i = \phi_a/A_{i(\text{net})}$ locates H_i on the B–H curve.

(2) Calculate $H_al_a = (B_a/\mu_0)l_a$ for the gap (where $B_a = \phi_a/A_a$), equation (19) being employed to find A_a.

(3) Required value of $NI = (H_il_i + H_al_a)$ by the circuital law of magnetism; it is assumed, of course, that the H's are expressed in ampere-turns per unit length and that the l's are expressed in these same unit lengths.

The problem of finding ϕ_a for a specified value of NI (of the exciting coil) is not a straightforward problem, but it can usually be solved by means of three or four well-chosen sets of calculations. The calculations must be made with a view toward satisfying two conditions:

$$(1)\ \phi_a = \phi_i \quad \text{and} \quad (2)\ H_il_i + H_al_a = NI \quad \text{(specified)} \tag{20}$$

Either trial-and-error methods may be employed to satisfy these conditions, or the graphical method shown in Fig. 14 may be employed. The graphical method shown in Fig. 14 is the same as that which is widely used in vacuum-tube circuits where the tube is the non-linear element and a fixed resistor in series with the tube is the linear element. In the present case, the core is the non-linear element (since ϕ_i is not directly proportional to mmf_i), and the air gap is the linear element (since ϕ_a is proportional to mmf_a).

Since the graphical method shown in Fig. 14 can be used to advantage

wherever non-linear circuit elements are involved, this method will be employed in meeting the two conditions stated in (20).

(1) Lay off the specified value of NI along the mmf axis of Fig. 14 and recognize that H_il_i (measured from 0) plus H_al_a (measured backward or to the left from the NI point) must equal NI.

(2) Arbitrarily select any value of ϕ_p for the purpose of finding one point on the linear ϕ versus H_al_a graph. Then

$$(H_al_a)_{\text{for } p} = \left(\frac{\phi_p}{\mu_0 A_a} l_a\right)_{\text{for point } p}$$

(3) Plot point p in Fig. 14 by measuring $(H_al_a)_{\text{for } p}$ to the left from the NI ordinate along the ϕ_p abscissa as shown. This point and the

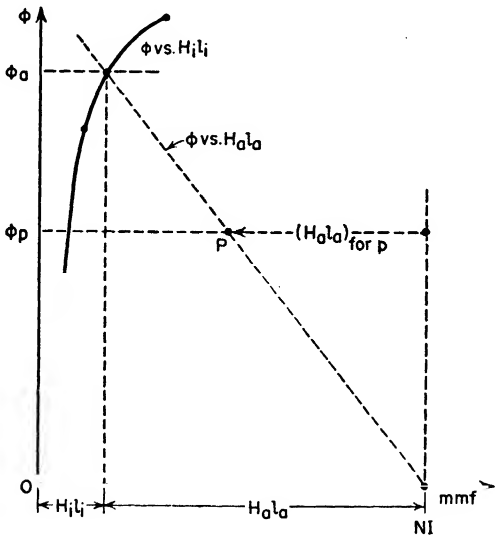


FIG. 14. A method for finding ϕ_a when NI is specified.

point (NI, 0) determine the straight ϕ versus H_al_a graph shown in Fig. 14.

(4) Determine H_il_i for three or four *assumed* values of ϕ such that the ϕ versus H_il_i points thus determined will, when plotted, either result in the ϕ versus H_il_i curve intersecting the ϕ versus H_al_a line or be close enough to this straight line so that by extrapolation the point of intersection can be found.

An examination of Fig. 14 will show that the point of intersection of the ϕ versus H_il_i curve and the ϕ versus H_al_a straight line meets both requirements stated in (20) above, and therefore ϕ_a, the air-gap flux,

for a specified value of NI is determined. In certain cases it may be simpler to solve the problem by trial-and-error methods, but the method shown in Fig. 14 is useful here in that it shows clearly what these trial-and-error calculations are attempting to accomplish.

Example. Let it be required to find the magnetic flux crossing the air gap of Fig. 13 if a 200-turn coil carrying a current of 1.5 amp energizes the magnetic circuit which is known to consist of

$$l_i = 30 \text{ in.} \quad l_a = 0.03 \text{ in.}$$

$$A_{\text{gross}} = 3 \times 4 \text{ sq in. core}$$

$$A_i = 0.9 \times 12 = 10.8 \text{ sq in.}$$

The core material is that for which curve E of Fig. 11 is applicable.

(1) In this case NI of Fig. 14 is $200 \times 1.5 = 300$ amp-turns.

(2) We *arbitrarily* select $\phi_p = 200{,}000$ maxwells in order to determine p.

$$A_a = (3.03)(4.03) = 12.2 \text{ sq in.} \quad (\text{or } 78.5 \text{ sq cm})$$

$$B_a = \frac{200{,}000}{78.5} = 2550 \quad \text{maxwells/sq cm} \quad (\text{or gauss})$$

$$H_a = \frac{B_a}{\mu_0} = 2550 \quad \text{gilberts/cm} \quad (\text{or } 5150 \text{ amp-turns/in})$$

since, for maxwells per square centimeter and gilberts per centimeter, $\mu_0 = 1$.

(3)

$$(H_a l_a)_{\text{for } p} = 5150 \times 0.03 = 155 \quad \text{amp-turns}$$

which is laid off to the left from $NI = 300$ amp-turns in Fig. 14 along the arbitrarily selected $\phi_p = 200{,}000$ maxwells line. Point p and (300, 0) determine the straight line graph of ϕ versus $H_a l_a$, and this graph when read *from* the NI point tells us how much of the total mmf is being consumed in the air gap.

(4) We now calculate one point on the ϕ versus $H_i l_i$ curve shown in Fig. 14 for an assumed value of $\phi = 300{,}000$ maxwells.

$$B_i = \frac{300{,}000}{10.8 \times 6.45} = 4300 \text{ maxwells/sq cm} \quad (\text{or gauss})$$

$$H_i = 0.8 \text{ gilbert/cm} \quad (\text{or } 1.6 \text{ amp-turn/in.}) \quad \text{from } B\text{–}H \text{ curve}$$

$$H_i l_i = 1.6 \times 30 = 48 \quad \text{amp-turns}$$

At $\phi = 300{,}000$ maxwells the air-gap magnetic potential drop will be

$$\frac{3}{2} \times 155 \doteq 233 \quad \text{amp-turns}$$

as determined from Fig. 14, if the plot is actually constructed, or by proportion, employing the result obtained for the air gap at $\phi = 200{,}000$ maxwells in (3) above.

Since 48 and 233 amp-turns do not add to 300 we know that the assumed value of 300,000 maxwells is somewhat too low, so we repeat the above calculations for an assumed ϕ of 325,000 maxwells and find no readable change in H_i (due to the shape of the B–H curve). Therefore

$$H_il_i = 48 \text{ amp-turns} \quad \text{(essentially the same as before)}$$

$$H_al_a = \frac{325}{200} \times 155 = 252 \quad \text{amp-turns}$$

Since $H_il_i + H_al_a = 300$ amp-turns, the problem is solved, and the result is $\phi_a = 325{,}000$ maxwells.

11. Parallel Magnetic Circuits. For the case of two independent magnetic circuits energized with the same mmf coil of NI amp-turns as

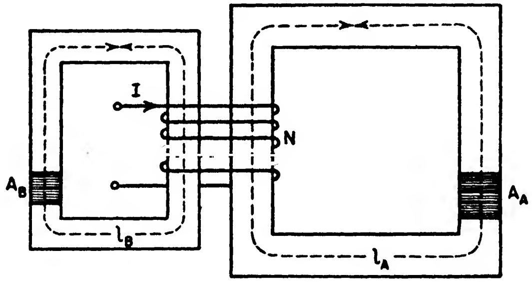


FIG. 15. Parallel magnetic circuits.

shown in Fig. 15, the two circuits are in parallel with respect to the common mmf source. Each of the two parallel branches is a series magnetic circuit which must satisfy the basic condition of all closed paths in a magnetic field, namely,

$$\Sigma Hl \text{ (around each closed path)} = \text{mmf} \tag{21}$$

If the mmf is given, the flux established in each of the two parallel circuits shown in Fig. 15 is found by the methods which are applicable to series circuits.

The only problem which arises in connection with an arrangement of the kind shown in Fig. 15 which might require a simultaneous solution of the two parallel circuits would be: "given the total flux linking the coil, find the mmf." To solve a problem of this kind, one may plot the ϕ_A versus mmf and the ϕ_B versus mmf curves as shown in Fig. 16, employing arbitrarily selected values of ϕ. Then ϕ_A and ϕ_B *at the same*

mmf may be added to obtain the $(\phi_A + \phi_B)$ versus mmf curve. In this way the mmf, NI, required to produce any assigned value of $\phi_t = \phi_A + \phi_B$ may be read directly from the upper graph shown in Fig. 16.

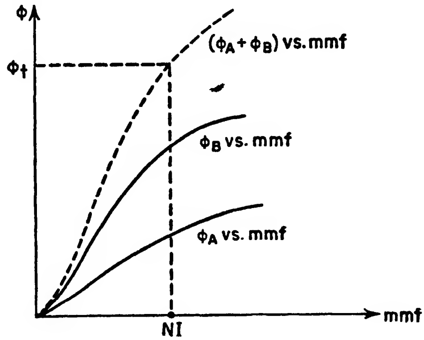


FIG. 16. Combining fluxes ϕ_A and ϕ_B to obtain $(\phi_A + \phi_B)$ versus mmf curve.

12. Series-Parallel Magnetic Circuits. The arrangement shown in Fig. 17 may be considered to be formed of two parallel branches, l_A and l_B, since between points M and N there must exist the same magnetic potential drop, that is,

$$\Sigma(Hl)_A = \Sigma(Hl)_B \tag{22}$$

The magnetic potential drop between M and N may be considered to be in series with the magnetic potential drop along the l_C path, and it

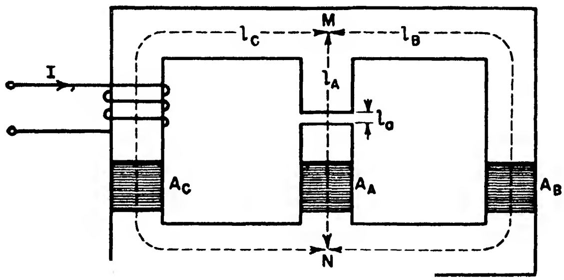


FIG. 17. Series-parallel magnetic circuit.

is plain from the circuital law of magnetism that the magnetic potential drop along the l_C path plus the magnetic potential drop from M to N will be equal to the mmf of the current-carrying coil which encircles the C leg. Thus

$$\Sigma(Hl)_C + \Sigma(Hl)_A = \Sigma(Hl)_C + \Sigma(Hl)_B = NI_{(\text{of mmf coil})} \tag{23}$$

Since the magnetic flux which threads through the l_C path divides at M between the l_A path and the l_B path, and since magnetic flux lines are continuous, it follows that

$$\phi_C = \phi_A + \phi_B \tag{24}$$

The three conditions stated in equations (22), (23), and (24) are analogous to the three basic conditions that would have to be satisfied simultaneously in a three-branch electrical network which had a battery E in place of NI and which had three resistances arranged as l_A, l_B, and l_C are arranged in Fig. 17. Equations (22) and (23) are analogous to Kirchhoff's voltage equations where the Hl's correspond to the RI's of the electrical circuit equations. Equation (24) is analogous to Kirchhoff's current law at a junction where the ϕ's correspond to the I's of the electrical network.

Regardless of how the problem is framed, any solution to the series-parallel magnetic circuit shown in Fig. 17 must satisfy the conditions stated in equations (22), (23), and (24).

Example. Let it be required to find the ampere-turns of the exciting coil in Fig. 17 when the flux through the l_A path is 100,000 lines under the following conditions:

$l_A = 10$ in. $l_B = 30$ in. $l_C = 30$ in.
Air-gap length in the l_A path $= 0.03$ in.
Gross cross-sectional area of the laminated core $= 3 \times 4$ sq in.
Net cross-sectional area of each path $= 0.9 \times 12 = 10.8$ sq in.

Except for the air gap l_a in the A path, the core material is assumed to be that for which the B–H curve shown in Fig. 18 applies.

The first step in the solution is plainly to find the drop in magnetic potential between points M and N. Employing series-circuit principles to path A, we add the magnetic potential drop in the iron $(H_il_i)_A$ to the drop in the air gap $(H_al_a)_A$ for the assigned 100,000 lines (or maxwells) in the l_A path, with the result:

$$(H_il_i)_A + (H_al_a)_A = 11 + 77 = 88 \quad \text{amp-turns} \quad (\text{drop } M\text{-to-}N)$$

From equation (22)

$$(Hl)_B = 88 \quad \text{amp-turns} \quad \text{or} \quad H_B = \frac{88}{30} = 2.9 \quad \text{amp-turns/in.}$$

$$B_B = 56.5 \text{ kilolines/sq in.} \quad (\text{from Fig. 18})$$

$$\phi_B = 56{,}500 \times 10.8 = 610{,}000 \quad \text{lines}$$

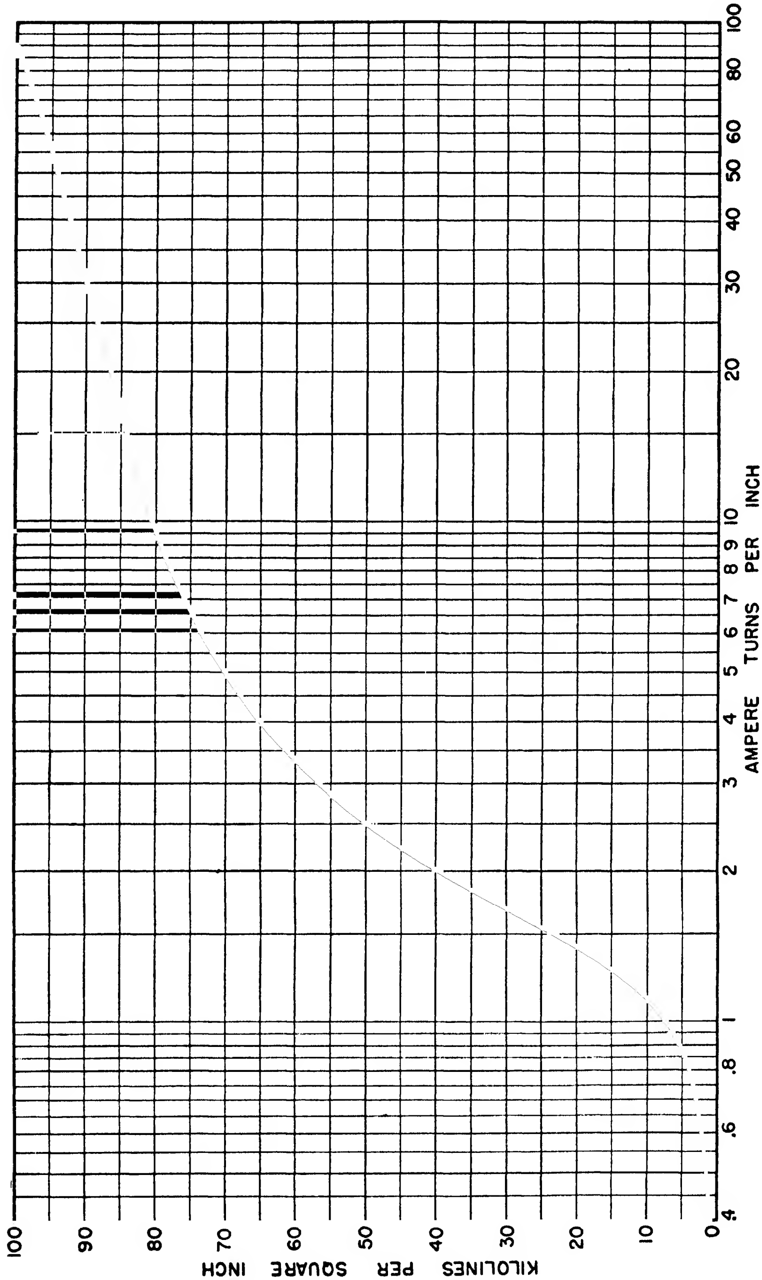


FIG. 18. *B–H* curve of Allegheny dynamo-grade steel sheets.

From equation (24),

$$\phi_C = \phi_A + \phi_B = 100{,}000 + 610{,}000 = 710{,}000 \quad \text{lines}$$

$$B_C = \frac{710{,}000}{10.8} = 66{,}000 \quad \text{lines/sq in.}$$

$$H_C = 4.2 \text{ amp-turns/in.} \quad \text{(from Fig. 18)}$$

$$(Hl)_C = 4.2 \times 30 = 126 \quad \text{amp-turns}$$

From equation (23),

$$\text{mmf of coil} = (Hl)_C + (Hl)_A = 126 + 88 = 204 \quad \text{amp-turns}$$

This example illustrates the ease with which magnetic circuit problems can be handled in terms of lines, inches, and square inches when the B–H curve is plotted as it is in Fig. 18.

13. Residual Magnetism. Thus far only the *rising* portion of a B–H curve has been considered. The experimental method for determining B versus H (as given in Section 7) presupposes that the magnetic material is carefully demagnetized before the incremental ΔH's are applied. The B–H curve then obtained is that which passes through $B = 0$, $H = 0$, as indicated by the curve labeled "increasing H_0" in Fig. 19.

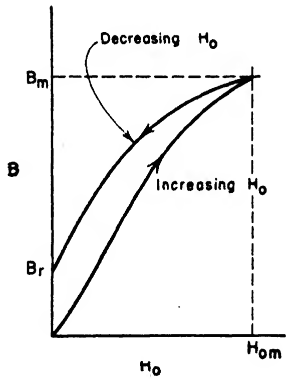


FIG. 19. Illustrating the retentivity of a magnetic material for the magnetism acquired.

After some maximum value H_{0m} in Fig. 19 is reached (say by way of incremental *increases* from zero), let it be assumed that H_0 is reduced to zero by way of incremental *decreases* in H_0. The result of decreasing H_0 is shown by the B–H curve labeled "decreasing H_0" in Fig. 19.

The two curves given in Fig. 19 show clearly that, once magnetized, the ferromagnetic material retains a *residual* flux density B_r which, if multiplied by the cross-sectional area of the material A, gives the *residual magnetism* or the magnetic flux retained by the material after the magnetizing current is reduced to zero.

The physical interpretation of residual magnetism is that, once the sub-atomic current loops are aligned, they tend at least in some degree to remain aligned and in so doing produce magnetism which is more or less permanent, depending upon the conditions under which the magnetization takes place and the molecular structure of the magnetic material.

14. Hysteresis Loss in Magnetic Cores. Most magnetic cores are energized with alternating current which causes the B—H relationship to take the form of a closed loop as shown in Fig. 20. This loop is called a *hysteresis loop.* Starting a cycle of events at $-H_m$ (due to $-I_m$), the loop is traced to $-B_r$ (when $H = I = 0$) up to $+H_m$ (due to $+I_m$)

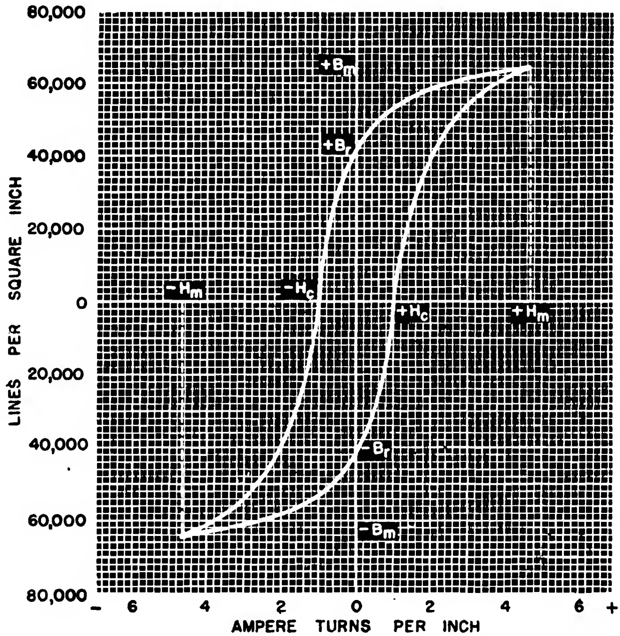


FIG. 20. Hysteresis loop of No. 29 gage U.S.S. dynamo-grade electrical steel sheets.

and back again to $-H_m$, whereupon the cycle is repeated in synchronism with the alternating current flowing in the mmf coil.

The magnitude of H required to overcome the previous value of B_r is called the *coercive force* and may be expressed in whatever units are used in expressing H. In Fig. 20, for example, the coercive force H_c required to nullify the effect of $B_r = 42{,}000$ lines/sq in. is 1 amp-turn/in. If higher values of B_m are employed than shown in Fig. 20, higher values of both B_r and H_c will accompany these higher maximum values of flux density. The higher the product $H_c \times B_r$, the more nearly the material approaches a permanent magnet, and the more energy is required to effect reversals of magnetism in the material.

In order to find the amount of energy required to carry the magnetism through the complete cycle of a hysteresis loop, we shall assume that voltage e in the following expression is the component of the applied

voltage which when combined with ($i\,dt$) will supply the energy which is necessary to effect a change of magnetism in the material. With this assumption we may write

$$\oint_{-I_m}^{-I_m} ei\,dt = \text{joules supplied per cycle of magnetization} \qquad (25)$$

where the limits and the small circle around the integral sign imply that we should sum ($ei\,dt$) over one cycle of the alternating current.

The voltage and current in equation (25) may be replaced with their equivalents in terms of B and H as shown below.

$$e_{\text{applied}} = N\frac{d\phi}{dt} = NA\frac{dB}{dt} \qquad \text{since } \phi = BA \qquad (26)$$

$$i = \frac{Hl}{N} \qquad \text{from } H = \frac{Ni}{l} \qquad (27)$$

Replacing e and i in equation (25) by these equivalents, we have

$$\oint_{-I_m}^{-I_m}\left(NA\frac{dB}{dt}\right)\left(\frac{Hl}{N}\right)dt = Al\oint_{-B_m}^{-B_m} H\,dB \qquad \text{joules/cycle (in mks units)} \qquad (28)$$

A study of Fig. 21 will show that, if B is treated as the independent variable,

$$\oint_{-B_m}^{-B_m} H\,dB = \text{area enclosed by the hysteresis loop} \qquad (29)$$

and, since $A \times l$ is the volume of the material,

$$\text{hysteresis loss} = (\text{area of loop}) \text{ joules/cu m/cycle} \qquad (30)$$

provided the area of the loop is measured in (webers/sq m) × (amp-turns/m).

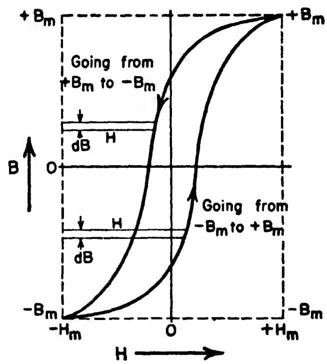


FIG. 21. Illustrating the $H\,dB$'s employed in calculating energy loss in terms of the area of a hysteresis loop.

Example. The area of the hysteresis loop shown in Fig. 20 is roughly 22.5 squares of

$$\left(10{,}000\,\frac{\text{lines}}{\text{sq in.}}\right)\left(1\,\frac{\text{amp-turn}}{\text{in.}}\right) = \left(\frac{10^4 \text{ maxwell} \times \text{amp-turns}}{\text{cu in.}}\right)$$

This energy per cubic inch converted to (weber × amp-turns)/cu m is

$$10^4 \times \frac{\text{webers}}{\text{maxwell}} \times \frac{\text{cu in.}}{\text{cu m}} = 10^4 \times 10^{-8} \times (6.07 \times 10^4)$$
$$= 6.07 \left(\frac{\text{weber} \times \text{amp-turns}}{\text{cu m}}\right)$$

The fact that the number 6.07 represents joules per cubic meter is plain if it is recognized that the weber is dimensionally volts × sec/turns, from

$$e = N\ (\Delta\text{webers}/\Delta\text{sec}) \quad \text{volts}$$

Thus the hysteresis loss is

$$W_h = \text{area of hysteresis loop} = 22.5 \times 6.07 = 136.6 \text{ joules/cu m/cycle}$$

At any particular frequency, say 60 cycles/sec, 1 cu m of the material considered in the above example would have a hysteresis power loss of

$$P_h = W_h f = 136.6 \times 60 = 8200 \quad \text{joules/sec} \quad (\text{or } 8200 \text{ watts})$$

This represents a sizable amount of power, but then a cubic meter of solid iron is a large amount of magnetic material.

15. Eddy-Current Loss in Magnetic Cores. Wherever there are reversals of magnetism in magnetic materials there is present in the material another type of loss (in addition to hysteresis loss). This second type of power loss is called *eddy-current* loss since it is caused by circulating currents which flow in the laminations due to the $d\phi/dt$ voltage which is generated within the iron itself. In order to keep this loss to a reasonable value, it is necessary to use the laminated type of cores which have been considered in the various magnetic circuit problems of this chapter.

Only two simple principles are required to show the manner in which the eddy currents within the iron cause Ri^2 or e^2/R losses in the iron. These principles are: $e = d\phi/dt$ and power $= e^2/R$, both of which are well known. The details of the derivation are, however, made somewhat involved by the number of factors which enter into the problem. The derivation will be outlined here in order to show that the eddy-current power loss per unit volume of the material varies as the square of the maximum flux density B_m^2, the square of the frequency f^2, and the square of the thickness of the lamination t^2. The result of the derivation shown below is

$$P_{\text{eddy}} = KB_m^2 f^2 t^2 \quad \text{watts} \tag{31}$$

where K depends upon the units employed and the resistivity of the iron, ρ. Since K varies inversely as ρ, manufacturers have produced silicon-type steel sheets which have high values of ρ and hence low values of eddy-current loss.

In most practical work, the designer uses information supplied by the manufacturers which specifies the combined hysteresis and eddy-current loss in different grades of sheet steel, usually in watts per pound per cycle at some particular value of B_m, the maximum flux density, at which the material is normally operated. In the literature this combined loss is often symbolized as P_{h+e}.

Derivation of Equation (31). In Fig. 22 is shown a cross-sectional view of one lamination of sheet steel that has an l dimension which is directed into the plane of the page. The magnetic flux is shown as being instantaneously directed *into* the page and, when the change-of-flux direction is *into* the page, the eddy currents circulate in the direction shown.

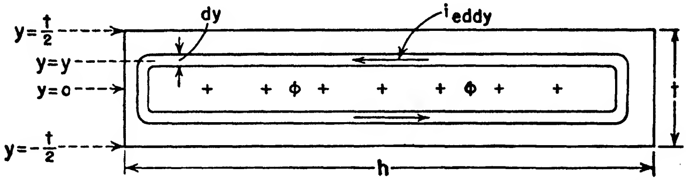


FIG. 22. Eddy current in a lamination which has a thickness t.

The voltage induced or generated in a loop which is bounded on the top and bottom by dy elements as shown in Fig. 22 produces the circulating currents that in turn result in heat power loss.

$$e \text{ (around the closed loop)} = \frac{d\phi}{dt} = A\frac{dB}{dt} \tag{32}$$

where the area A of the loop considered is about $2yh$.

If an alternating current is used to magnetize the material, and we assume a sinusoidal variation of this exciting current, we may express B in equation (32) as

$$B = B_m \sin (2\pi ft) \tag{33}$$

Then

$$e = AB_m(2\pi f) \cos (2\pi ft) = 2yhB_m 2\pi f \cos (2\pi ft)$$

$$e^2 = 16\pi^2 h^2 B_m{}^2 f^2 y^2 \cos^2 \alpha \quad (\text{where } \alpha = 2\pi ft)$$

Since $\cos^2 \alpha = \frac{1}{2} + \frac{1}{2}\cos 2\alpha$, we know that the *time-averaged* value of e^2 is

$$E_{\text{av.}}{}^2 = 8\pi^2 h^2 B_m{}^2 f^2 y^2 \quad \text{(over any whole number of cycles)} \tag{34}$$

The resistance of the loop through which the eddy current shown in Fig. 22 flows is

$$R = \rho \frac{l_e}{A_e} = \rho \frac{2h}{l\,dy} \tag{35}$$

where l_e is the length of the eddy-current path, about $2h$ in Fig. 22
A_e is the cross-section through which i_{eddy} flows, $l\,dy$
l is the length of the lamination in the flux direction.

The time-averaged value of eddy-current power loss is obtained by dividing $E_{\text{av.}}^2$, as given in equation (34), by the resistance of the eddy-current path R, as given in equation (35). Thus

$$dP_e \text{ (per loop)} = \frac{4\pi^2 h B_m{}^2 f^2 l y^2\,dy}{\rho} \tag{36}$$

$$\begin{aligned} P_e \text{ (per lamination)} &= \frac{4\pi^2 h l B_m{}^2 f^2}{\rho} \int_{-t/2}^{t/2} y^2\,dy \\ &= \frac{4\pi^2 h l B_m{}^2 f^2}{\rho} \left[\frac{y^3}{3}\right]_{-t/2}^{t/2} \\ &= \frac{(hlt)\pi^2 B_m{}^2 f^2 t^2}{3\rho} \end{aligned} \tag{37}$$

Since the volume of the lamination considered is hlt, the eddy-current power loss is

$$P_e = \frac{\pi^2 B_m{}^2 f^2 t^2}{3\rho} \quad \text{watts/cu m} \quad \text{(in mks units)} \tag{38}$$

where B_m is the maximum value of the flux density in webers per square meter
f is the frequency of the exciting current in cycles per second
t is the thickness of the lamination expressed in meters
ρ is the resistivity of the magnetic material expressed in ohms per cubic meter or, more precisely, (ohm $\times$ square meters) per meter.

16. Summary. The magnetic flux in iron cores and associated air gaps has been evaluated in terms of simple circuit concepts employing the single basic relationship

$$\Sigma Hl \text{ (around a loop)} = \text{mmf (around the loop)} \tag{39}$$

with due regard for the fact that magnetic flux lines are continuous and that they *spread out* or fringe at the air gaps.

In manipulating equation (39), it is necessary to use experimentally determined B–H curves which may come to us expressed either in secondary units of B and H or in ab-cgs units where the maxwell per square centimeter (or gauss) is the primary unit of flux density and the gilbert per centimeter (or oersted) is the primary unit of magnetizing force or magnetic potential gradient.

Ohm's law of magnetic circuits is

$$\phi = \frac{\text{mmf}}{\mathcal{R}} \quad \text{or} \quad Hl = \text{mmf} \tag{40}$$

where $\mathcal{R} = \dfrac{l}{\mu_0 \mu_r A}$ is called the reluctance of the magnetic path

$\mu_r = \dfrac{B}{\mu_0 H}$ is called the relative permeability of the path.

Since μ_r is distinctly variable with respect to H (or B), the concept of reluctance is of little use except possibly from the point of view of analogy.

Where magnetic cores are energized with alternating current two distinctly different forms of power loss occur in the core material:

$$\text{hysteresis power loss } P_h = (\text{area of hysteresis loop})\,(f) \quad \text{watts/cu m} \tag{41}$$

provided the area is expressed in (watt-sec/cu m) = (webers/sq m) times (amp-turns/m) and f is expressed in cycles per second.

$$\text{eddy-current power loss } P_e = \frac{\pi^2 B_m{}^2 f^2 t^2}{3\rho} \quad \text{watts/cu m} \quad \text{[See (38).]}$$

PROBLEMS

1. What is the relative permeability of a piece of steel which, if subjected to a magnetizing force of 40 amp-turns/m, possesses a flux density of 0.01257 weber/sq m?

2. If an **H** vector enters an iron surface from an air gap at an angle of 80° (to the surface), what *change* in angle from its air-gap direction is experienced by the **H** vector at the boundary surface? (μ_r of the iron = 100.)

3. If an **H** vector enters an iron surface from an air gap at an angle of 80° (to the surface) with an air-gap magnitude of 400 amp-turns/m, what is the magnitude of this **H** vector just inside the iron? (μ_r of the iron = 100.)

4. Refer to Fig. 7, page 286, and assume that the pole surface (ab) is maintained at a magnetic potential difference of 4000 amp-turns with respect to the top of the teeth by current-carrying coils which are not shown in the magnetic field map.

(*a*) What is the magnitude of the magnetic potential gradient expressed in ampere-turns per meter in the air gap between the pole and the tooth center (along line ac or line bd in Fig. 7) if the distance ac is 1 cm?

(*b*) What is the magnitude of the **H** vector just inside the surface of the iron tooth at point *c* in Fig. 7?

(*c*) At what place on the field map is the highest potential gradient indicated and what is the approximate magnitude of this maximum value?

5. If the pole surface (*ab*) in Fig. 7 is maintained at a magnetic potential difference of 4000 amp-turns relative to the top of the iron teeth and the length of air gap (*ac*) is 1 cm, what is the magnetic flux density at the point *c* which is at the middle of a tooth?

6. Refer to Fig. 8, page 287. Let it be assumed that $r = 20$ cm and $d = 4$ cm.

(*a*) What is the reluctance (expressed in ampere-turns per weber) if the core is non-ferromagnetic, that is $\mu_r = 1$?

(*b*) What amount of magnetic flux is established in this core if the distributed winding of 1200 turns carries 0.40 amp?

7. Given $B = 35{,}500$ lines/sq in., express B in webers per square meter and in maxwells per square centimeter.

8. Given $H = 202$ amp-turns/in., express H in ampere-turns per meter and in gilberts per centimeter.

9. Plot a graph showing the variation in μ_r of annealed Allegheny electric metal versus H in gilberts per centimeter from $H = 0.04$ to $H = 3.6$ gilberts/cm. (See Fig. 11, page 292.)

10. Plot a graph showing the variation in μ_r of dynamo-grade steel sheets versus B in kilolines/sq in. from $B = 2$ to $B = 100$ kilolines/sq in. (See Fig. 18, page 304.)

11. Plot the B–H curve for the magnetic core shown in Fig. 10, page 290, from the following data:

$$r = 20 \text{ cm} \quad a = 4 \text{ cm} \quad b = 5 \text{ cm} \quad N = 100 \text{ turns} \quad N_g = 5 \text{ turns}$$

The galvanometer sensitivity K is known to be 10,000 maxwell-turns per scale division.

After the magnetic core has been carefully demagnetized ($B = 0$, $H = 0$), the following incremental changes in the magnetizing current I are observed to produce galvanometer deflections D as shown below.

Step	ΔI amp	I after change	D scale div.	Step	ΔI amp	I after change	D scale div.
1	0.2	0.2	4.0	7	0.5	2.0	15.0
2	0.2	0.4	6.0	8	1.0	3.0	15.0
3	0.2	0.6	10.0	9	1.0	4.0	9.0
4	0.2	0.8	10.0	10	3.0	7.0	10.0
5	0.2	1.0	20.0	11	3.0	10.0	7.0
6	0.5	1.5	25.0	12	8.0	18.0	10.0

Plot B (in maxwells per square centimeter) versus H (in gilberts per centimeter) from the above data.

12. Plot the B–H curve of Prob. 11 employing B (in lines per square inch) and H (in ampere-turns per inch).

13. Refer to Fig. 10. It will be assumed here that:

$$r = a' = b = 4 \text{ cm} \quad NI = 600 \text{ amp-turns} \quad \mu_r = (100x_{cm} + 800)$$

That is, μ_r is assumed to vary linearly from a value of 1000 at $x = 2$ cm to a value of 1400 at $x = 6$ cm owing to the change in H from

$$H = \frac{600}{4\pi} \text{ amp-turns/cm} \quad (\text{or } H = 60 \text{ gilberts/cm}) \quad \text{at } x = 2 \text{ cm}$$

to

$$H = \frac{600}{12\pi} \text{ amp-turns/cm} \quad (\text{or } H = 20 \text{ gilberts/cm}) \quad \text{at } x = 6 \text{ cm}$$

(*a*) Determine the approximate amount of flux in the core, basing calculations on the value of B determined at the center of the core ($x = r$), as $\phi = B_r A = \mu_0 \mu_{rr} H_r A$.

(*b*) Determine the correct value of flux in the core on the basis of

$$\phi = \int d\phi = \int_2^6 B_x (b\, dx)$$

14. A magnetic circuit which is homogeneous along the l dimension, like that in Fig. 12, page 296, is composed of annealed Allegheny electric metal laminations, curve D of Fig. 11. The length of the circuit is 20 cm, the gross cross-sectional area is (1.5×2) sq in., and the stacking factor is 0.88.

Find the flux established in the core by an exciting coil of 80 turns carrying 0.5 amp.

15. A magnetic circuit with an air gap in the l dimension, like that in Fig. 13, page 298, has the dimensions

$$l_i = 76 \text{ cm} \quad l_a = 0.0762 \text{ cm} \quad A_{\text{gross}} = (3 \times 4) \text{ sq in.} \quad \text{S.F.} = 0.9$$

The core material is transformer-grade steel sheets, curve E of Fig. 11.

Find the current required to establish 500,000 maxwells of air-gap flux if the magnetizing coil is composed of 200 turns.

16. The magnetic circuit described in Prob. 15 is energized with a 400-turn coil which carries 1.25 amp. Find the magnetic flux which crosses the air gap.

17. Find the core flux in a magnetic circuit like that shown in Fig. 13 if an 890-turn coil carrying 10 amp is used to energize the circuit. The known data are

$$l_i = 30 \text{ in.} \quad l_a = 0.30 \text{ in.} \quad A_{\text{gross}} \doteq (3 \times 4) \text{ sq in.} \quad \text{S.F.} = 0.9$$

The core material is dynamo-grade steel sheets; B–H curve shown in Fig. 18, page 304.

18. Find the total flux linking with the magnetizing coil of Fig. 15, page 301, if this coil has an mmf of 300 amp-turns and the core specifications are

$$l_A = 60 \text{ in.} \quad A_{A\,\text{net}} = 10.8 \text{ sq in.} \quad l_B = 30 \text{ in.} \quad A_{B\,\text{net}} = 6 \text{ sq in.}$$

Applicable B–H curves: curve D of Fig. 11 for the A circuit material; curve E of Fig. 11 for the B circuit material. Both circuits are homogeneous in the direction of the flux paths.

19. Find the mmf required to establish a total flux of 0.011 weber through the magnetizing coil of Fig. 15 if the cores have the following specifications:

$$l_A = 30 \text{ in.} \quad A_{A\,\text{net}} = 10.8 \text{ sq in.} \quad l_B = 20 \text{ in.} \quad A_{B\,\text{net}} = 6 \text{ sq in.}$$

Applicable B–H curve: Fig. 18, page 304, for both circuits. Circuits are both homogeneous in the direction of the flux paths.

20. Refer to Fig. 17, page 302. Find the mmf of the coil to produce an air-gap flux (in the A path) of 76,000 maxwells under the following conditions:

$$l_A = 10 \text{ in.} \quad l_B = l_C = 30 \text{ in.} \quad \text{length of air gap } l_a = 0.03 \text{ in.}$$

The net cross-sectional area of the core, $A_A = A_B = A_C = 10.8$ sq in. Applicable B–H curve: Fig. 18, page 304, throughout. $A_{\text{gap}} = 12$ sq in.

21. What flux will be established in each of the three paths (l_A, l_B, and l_C) of Fig. 17, page 302, by a 100-turn coil carrying 1.5 amp (wound on the C leg as shown) if the core data and the air-gap length are the same as given in Prob. 20.

22. Find the mmf *per pole* required to produce a useful air-gap flux ϕ_M of 1200 kilolines/pole in Fig. 23 if the leakage flux per pole is $0.2\phi_M$ or 240 kilolines. The specifications are

Member	Length on a Per Pole Basis (in.)	Net Cross-Sectional Area (sq in.)	Flux Present in Member (kilolines)	Material	Applicable B–H Curve
Pole yoke	8	2.5 × 5	720	Rolled steel	Use Fig. 18
Pole	5	3.5 × 4.5	1440	Cast steel	Curve E, Fig. 11
Armature	2.5	8.1	600	Steel sheets	Fig. 18
Air gap	0.1	16.55	1200		

The net area of the armature is obtained by using a stacking factor of 0.9, that is, 0.9(2 × 4.5) sq in., where the 4.5-in. dimension is normal to the plane of the page in Fig. 23. The air-gap area is obtained by making the customary allowance for fringing at short air gaps.

23. After $I = 18$ amp is reached in Prob. 11, the following incremental *decreases* in magnetizing current and corresponding galvanometer deflections are obtained for the decreasing B–H curve:

Magnetizing Current after Change from 18 amp	Galvanometer Deflections which Accompany the ΔI's
10 amp	− 6 scale div.
4	−15
2	−15
1	−20
0	−30

Determine the residual flux density in the material by plotting the decreasing B–H curve from $H = 1800/40\pi$ amp-turns/cm (or 18 gilberts/cm) where $B = 14{,}100$ maxwells/sq cm, from Prob. 11, to $H = 0$.

24. The area of a hysteresis loop is measured and found to represent

$$100\left(\frac{\text{webers}}{\text{sq m}} \times \frac{\text{amp-turns}}{\text{m}}\right) = 100 \text{ watt-sec/cu m/cycle}$$

What is the hysteresis power loss in a 90-cu cm net volume of the material working under the conditions for which this hysteresis loop is applicable if the frequency is 440 cycles/sec?

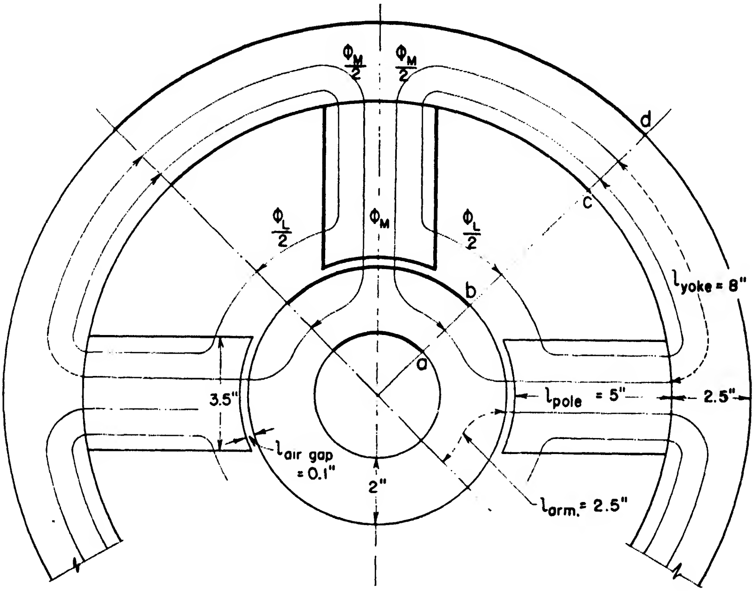


FIG. 23. The magnetic circuit of most d-c machines and synchronous (a-c) machines. See Prob. 22.

25. The area of a hysteresis loop is measured and found to represent

$$30{,}000\left(\frac{\text{maxwells}}{\text{sq cm}} \times \frac{\text{gilberts}}{\text{cm}}\right) = 2390 \text{ ergs/cu cm/cycle}$$

(*a*) What is the hysteresis power loss in watts per cubic centimeter at a frequency of 60 cycles/sec?

(*b*) If the density of the material is 7.5 (weighs 7.5 g/cu cm), what is the hysteresis loss expressed in watts per pound per cycle?

26. Refer to the hysteresis loop shown in Fig. 20 where B_m = 64,500 lines/sq in. or 1 weber/sq m. It has been shown that, for this particular value of B_m, the hysteresis loss is about 8200 watts/cu m at 60 cycles/sec.

Determine the eddy-current power loss at 60 cycles/sec, at B_m = 1 weber/sq m, in this material by means of equation (38) if the thickness of laminations is 0.04 cm and the resistivity of the material is thrice that of pure iron, namely

$$\rho = 30 \times 10^{-6} \text{ ohms/cu cm} \quad \text{or} \quad \rho = 30 \times 10^{-8} \text{ ohms/cu m}$$

27. The combined hysteresis and eddy-current power loss in a specimen of laminated steel sheets at 10,000 gauss at 30 cycles/sec is 0.5 watt/lb, and at 10,000 gauss at 60 cycles/sec the combined loss is 1.3 watts/lb.

(*a*) What is the hysteresis component of the core loss at 60 cycles/sec?

(*b*) What is the eddy-current component at 60 cycles/sec?

Note: Since B_m is the same, $P_{h+e} = K_h f + K_e f^2$ at either frequency.

28. What are the hysteresis and eddy-current components of the total core loss in Prob. 27 at 30 cycles/sec?

29. Refer to Fig. 8, page 287. What is the value of H at the center of the circle which has a periphery of $2\pi r$ expressed as a function of I and r? Specify the direction of **H** if the top plate of the battery is positive.

CHAPTER XII

Inductance

1. Concept and Definition of Inductance L. Wherever an electrical current or charge in motion exists, a magnetic field which links with the path of the current exists. Since the magnetic field is the outward manifestation, in space, of the kinetic energy of the moving charge, this kinetic energy may be thought of as existing or residing *in the magnetic field*.

For steady or non-time-varying currents, the magnetic field does not react upon the electrical circuit which produces it; but whenever the current changes in magnitude, the kinetic energy which is associated with the moving charge or current reacts upon the electrical circuit, demanding additional energy from the system if the current is increased and returning energy to the system if the current is decreased in magnitude. The accepted method of accounting for these energy transfers is in terms of a circuit parameter called self-inductance which is symbolized by L.

The self-inductance of any current-carrying circuit is defined as the *flux linkages per unit current* which in mks units means the number of weber-turns of flux linkage established by 1 amp of current flowing in the circuit; that is,

$$L = \frac{N\phi}{I} \qquad \frac{\text{weber-turns}}{\text{amp}} \quad \text{or henrys} \tag{1}$$

where, as will be shown presently, L depends only upon the dimensions of the circuit turns, the number of turns, and the relative permeability of the region in which the magnetic flux linkages occur.

Equation (1) may be arranged to read $LI = N\phi$, and for time-varying currents and magnetic fluxes to read

$$Li = N\phi \tag{2}$$

Of the various analogies which may be drawn between electrical and mechanical systems, the simplest and most instructive is probably that in which inductance is made analogous to mass. In this analogy Li is

to an electromagnetic system what mv (momentum) is to a mechanical system, and it will be remembered that the time derivative of momentum in a mechanical system represents the *inertial* force which tends to resist any change in velocity v of the moving mass m. The time derivative of electromagnetic momentum Li,

$$\frac{d(Li)}{dt} \quad \frac{d(N\phi)}{dt} = -e_{si} \tag{3}$$

is called the voltage of self-inductance and the minus sign in equation (3) indicates that e_{si} acts around the electrical circuit in a direction which tends to oppose the change of flux (or of current) which is taking place.

It will be observed that e_{si} in equation (3) is a voltage which is generated in the electrical circuit by the time rate of change of circuit current itself, and if L is a constant and ϕ is well defined (all ϕ linking with N turns) equation (3) becomes

$$L\frac{di}{dt} = N\frac{d\phi}{dt} = -e_{si} \tag{3-a}$$

This countervoltage of self-inductance must be taken into account whenever i is time-varying as, for example, in a simple resistive type of circuit which is energized with a time-varying driving voltage e. In this case the voltage of self-inductance enters into a Kirchhoff voltage equation as a *back* or countervoltage in the same manner as the Ri voltage drop; that is,

$$L\frac{di}{dt} + Ri = e \quad \text{or} \quad N\frac{d\phi}{dt} + Ri = e \tag{4}$$

In other words, $L(di/dt)$ or $N(d\phi/dt)$ is a component voltage drop in the circuit which is caused by the time-varying current and which must be reckoned with whenever (di/dt) or $(d\phi/dt)$ is not zero.

The fact that self-inductance L is a measure of the energy stored in the magnetic field may be shown as follows. The instantaneous power delivered to the system by the generator is ei, and

$$\left(L\frac{di}{dt}\right)i + (Ri)i = ei \tag{5}$$

is a power equation which, if considered over any period of time (Δt), may be used to form the energy equation which reads

$$\underbrace{L\int_0^{\Delta t} i\frac{di}{dt}\,dt}_{\substack{\text{kinetic energy}\\ \text{to magnetic field}}} + \underbrace{R\int_0^{\Delta t} i^2\,dt}_{\substack{\text{heat energy}\\ \text{developed in } R}} = \underbrace{\int_0^{\Delta t} ei\,dt}_{\substack{\text{energy supplied}\\ \text{by generator } e}} \tag{6}$$

The kinetic energy stored in the magnetic field in bringing the current from zero value to some fixed value I is, from the first term of equation (6),

$$L\int_0^I i\,di = L\left[\frac{i^2}{2}\right]_0^I = \frac{LI^2}{2} \quad \text{joules} \quad \text{(in mks units)} \tag{7}$$

The *change* in stored energy or the energy transfer between the magnetic field and the electrical circuit for a change of current from I_1 to I_2 is

$$L\int_{I_1}^{I_2} i\,di = L\left[\frac{i^2}{2}\right]_{I_1}^{I_2} = \frac{L(I_2{}^2 - I_1{}^2)}{2} \quad \text{joules} \tag{8}$$

Since $L(di/dt) = N(d\phi/dt)$, the above concepts might have been presented with equal facility in terms of the $N(d\phi/dt)$ voltage, but since the circuit parameter L can be readily measured in terms of standard inductances it is customary practice to employ the $L(di/dt)$ form as the voltage of self-inductance.

2. A Standard of Self-Inductance. Wherever the flux linkages per unit current can be evaluated, the value of L follows directly from equation (1). The flux linkages in the ring-shaped core shown in Fig. 1 can be evaluated to a high degree of accuracy if the magnetizing turns are of small diameter, uniformly distributed, and tightly packed along the entire circumference as shown in Fig. 10, page 290.

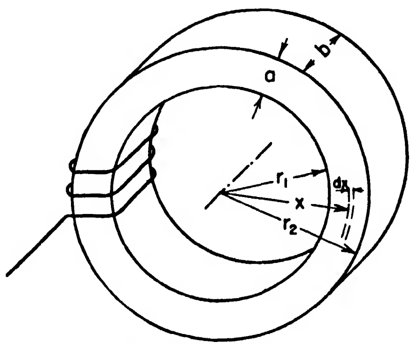


FIG. 1. Ring-shaped core of cross-section $a \times b$.

Since H_x in the core of Fig. 1 is $(NI/2\pi x)$ amp-turns/m, the flux density at distance x from the center (on the basis of $\mu_r = 1$) is

$$B_x = \mu_0 H_x = \mu_0 \frac{NI}{2\pi x} = 2 \times 10^{-7} \frac{NI}{x} \quad \text{webers/sq m} \tag{9}$$

The magnetic flux which crosses an area $(b\,dx)$ is

$$d\phi_x = B_x(b\,dx) = 2 \times 10^{-7} \frac{NIb\,dx}{x} \tag{10}$$

and the entire core flux is

$$\phi_x = 2 \times 10^{-7} NIb \int_{r_1}^{r_2} \frac{dx}{x} = 2 \times 10^{-7} NIb \ln \frac{r_2}{r_1} \quad \text{webers} \tag{11}$$

Since this number of webers links with all N turns of the winding

$$N\phi = 2 \times 10^{-7} N^2 I b \ln \frac{r_2}{r_1} \quad \text{weber-turns} \tag{12}$$

and

$$L = \frac{N\phi}{I} = 2 \times 10^{-7} N^2 b \ln \frac{r_2}{r_1} \quad \text{henrys} \tag{13}$$

where the dimension b is expressed in meters.

Equation (13) shows that the inductance of the arrangement illustrated in Fig. 1 depends only upon the dimensions of the turns and the number of turns. If the core is made of a durable material (like marble) where the dimensions remain essentially fixed, the ring-shaped arrangement can be (and is) used as a standard inductance (or standard measure of flux-linkages per unit current). Then other circuit configurations where the flux linkages cannot be so easily evaluated may be compared with the standard (say in an a-c bridge) to obtain the self-inductance of these circuit configurations.

Example. If in the ring-shaped arrangement shown in Fig. 1,

$r_1 = 20$ cm $\quad a = 4$ cm $\quad b = 5$ cm $\quad$ and $\quad N = 4000$ turns of No. 30 wire

$$L = (2 \times 10^{-7})(4000^2)(0.05) \ln \tfrac{24}{20} = 0.16 \ln 1.2$$

$$= 0.16 \times 0.1823 = 0.029168 \quad \text{henry}$$

The five-significant-figure accuracy is unwarranted but is written here for purposes of comparison with a refinement which is suggested in Prob. 4 at the close of the chapter.

It will be observed that a coil of many turns of relatively large dimensions is required to obtain this fractional part of 1 henry of self-inductance. By using a ferromagnetic core, the inductance might be made thousands of times larger, but since μ_r of a magnetic material varies so widely with the magnitude of the exciting current this would not be advisable in the case of a standard inductance.

3. Self-Inductance of Long Lines. The self-inductance of either a parallel-wire transmission line or of a coaxial-cable line may be determined quite accurately by means of equation (1). In Fig. 2 is shown a view of a parallel-wire line, and it is plain from the right-hand rule that both conductors produce flux linkages with the electrical circuit which is formed by the two line wires. This same arrangement is shown in cross-section in Fig. 9, page 207, and the method of calculating the **H** field in the vicinity of these two wires by the principle of superposition has already been considered in some detail.

If it is assumed that the region has a relative permeability of unity, the flux density *between* the two conductors shown in Fig. 2, due to current in the A conductor, is

$$B_x = \mu_0 H_x = \mu_0 \frac{I}{2\pi x} = 2 \times 10^{-7} \frac{I}{x} \quad \text{webers/sq m} \tag{14}$$

The magnetic flux which crosses a rectangular area bounded on one side by dx and on the other side by 1 m of axial length of line is

$$d\phi_x = B_x\,dA = B_x(1 \times dx) = 2 \times 10^{-7} \frac{I\,dx}{x} \qquad r < x < (D - r)$$

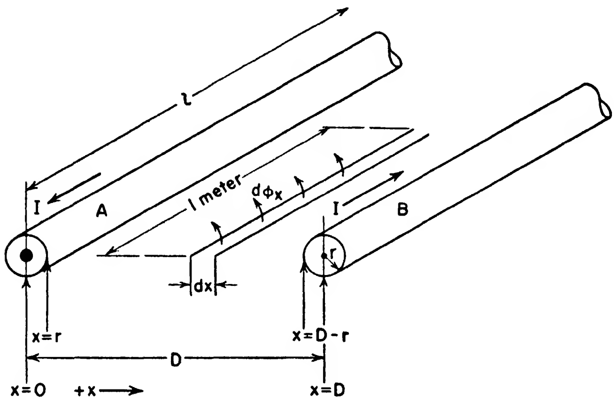


FIG. 2. A view of a two-wire line showing $d\phi_x$ due to current in the A conductor.

The external flux linkage produced by the current in the A conductor per meter length of line is

$$\begin{aligned} N\phi = \phi_x\Big]_{x=r}^{x=D-r} &= 2 \times 10^{-7} I \int_r^{D-r} \frac{dx}{x} \\ &= 2 \times 10^{-7} I \ln \frac{D-r}{r} \quad \text{weber-turns} \end{aligned} \tag{15}$$

The flux linkage produced by the current in the B conductor is, by symmetry, precisely the same as that given in equation (15) and is directly additive when considered across the area which is 1 m long and $D - 2r$ wide in Fig. 2.

Hence the inductance of the parallel-wire line *per meter length* of line due to flux linkages which are external to the surfaces of the conductors is

$$L_{\text{ext.}} = \frac{N\phi}{I} = \frac{2\phi_x}{I} = 4 \times 10^{-7} \ln \frac{D-r}{r} \quad \text{henrys/m} \tag{16}$$

Partial flux linkage within the conductors themselves also contribute to the total self-inductance of the parallel-wire line in Fig. 2. Where $x < r$ in Fig. 2,

$$H_x = \frac{\frac{x^2}{r^2} I}{2\pi x} = \frac{Ix}{2\pi r^2} \qquad \text{[See equation (9), page 208.]}$$

Hence for 1-m length inside the A conductor

$$d\phi_x = B_x\,dx = \mu_0 \frac{Ix}{2\pi r^2}\,dx \qquad 0 < x < r \tag{17}$$

This flux $d\phi_x$ links with only $\pi x^2/\pi r^2$ of the total circuit current, if uniform current density is assumed. Since a weber of flux linking with the entire circuit current is our basis of counting flux linkages, we must treat the internal flux which links with only x^2/r^2 of the total current on the basis of this partial linkage. This can readily be done by assigning to each unit of internal flux x^2/r^2 of a complete turn rather than give to this flux the status of a full turn of linkage. Thus

$$d(N_x\phi_x) = \frac{\mu_0 I}{2\pi r^2}\left(\frac{x^2}{r^2}\right) x\,dx \quad \text{(for flux inside the wire)}$$

The internal flux linkages of conductor A per meter length of conductor is

$$N_x\phi_x\Big]_{x=0}^{x=r} = \frac{2 \times 10^{-7}}{r^4} I \int_0^r x^3\,dx$$

$$= 2 \times \frac{10^{-7}}{r^4} I \left[\frac{x^4}{4}\right]_0^r = \frac{10^{-7} I}{2} \quad \text{weber-turns}$$

and the self-inductance which is caused by this internal linkage in one conductor is

$$L_{\text{int.}} \text{ per wire} \doteq \frac{10^{-7}}{2} \quad \text{henry/m} \tag{18}$$

The total low-frequency inductance of the two-wire line shown in Fig. 2 is

$$L = L_{\text{int.}} + L_{\text{ext.}} = \left(1 + 4 \ln \frac{D - r}{r}\right) 10^{-7} \quad \text{henrys/m} \tag{19}$$

The internal portion of L, $L_{\text{int.}}$, decreases from the value shown in equation (19) at high frequencies because the current along the center axis of a conductor, in experiencing greater flux linkages, is reduced in magnitude relative to the current in the outside layers of the conductor

near $x = r$. These outer layers of conductor do not have as much flux linking with them as do the inner layers, and the lower value of L of these outer paths is responsible for higher current densities at the surface of the conductor than inside the conductor.

It will be observed in equation (19) that the $L_{\text{int.}}$ portion due to both wires is 10^{-7} henry/m regardless of the size of the conductors or the spacing of the conductors. It is this portion of the total inductance which decreases with increase of frequency and, at frequencies of 10^7 or 10^8 cycles/sec, $L_{\text{int.}}$ becomes so small that it can usually be neglected.

Example. Let it be required to find the self-inductance of a 3-km length of coaxial cable like that shown in Fig. 10, page 208, if

$$r = 0.5 \text{ cm} \quad r_i = 2.5 \text{ cm} \quad \mu_r = 1 \text{ throughout}$$

neglecting the flux linkages between $x = r_i$ and $x = r_0$ since the partial linkage in the outer conductor due to the oppositely directed B's in this region makes this component of flux linkage very small.

From the one-wire bases of derivation employed in connection with equations (15) and (18), it is plain that the self-inductance of the coaxial cable is just one-half of the value given by equation (19) if $D - r$ in this equation is replaced by r_i of the coaxial arrangement. Hence

$$L_{\text{coax.}} = l\left(0.5 + 2\ln\frac{r_i}{r}\right)10^{-7} \quad \text{henrys} \tag{20}$$

$$L_{\text{coax.}} = 3000(0.5 + 2\ln 5)10^{-7}$$
$$= 3000(0.5 + 2 \times 1.61)10^{-7} = 1.116 \times 10^{-3} \quad \text{henry}$$

from which it will be seen that, at low-frequencies, the internal inductance of the inner conductor represents 0.5/3.72 of the total self-inductance of the cable.

4. Approximate Methods Employed in Determining Self-Inductance of Coils. Since the flux density distribution over the face of a turn of wire is non-uniform and practically impossible to evaluate,[1] approximate methods are employed to determine the self-inductances of coils. In addition to the non-uniformity of the flux density over the face of one turn, another complicating factor arises as shown in Fig. 3. Some of the magnetic flux links with only a portion of the total number of turns of the coil, making the summation of $N\phi$ a difficult task.

By recognizing the two types of difficulties involved (non-uniformity of B and partial linkage), we can understand why, in using empirical

[1] Theoretically, Ampere's law might be applied to each point on the face of a coil and the resultant **B** determined, but the summation of the vector $d\mathbf{B}$'s, even at a single point which is not symmetrically located relative to all the current-carrying elements ($I\,d\mathbf{l}$'s), represents an impossible task.

formulas applicable to multiple-layer coils, it becomes necessary to employ two empirically determined factors.

It is by actually measuring the self-inductance of many coils, of various dimensions and of various numbers of turns and layers, that empirical equations have been established that are accurate to about

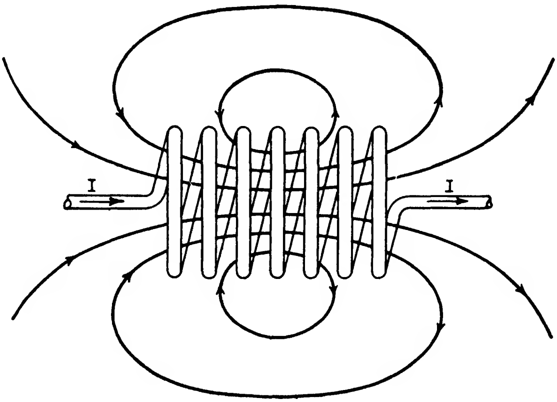


FIG. 3. Illustrating *partial* linkage of some flux paths and *total* linkage of others.

2 per cent over wide ranges of dimensions and numbers of turns. One of these empirical formulas which is based on an extension of a method given in Bureau of Standards Bulletin No. 1, Vol. 8, is

$$L = N^2 a\,(A - B)\,10^{-9} \quad \text{henrys} \tag{21}$$

where N is the number of turns
 a is the mean radius of the circular turns expressed in centimeters
 A and B are factors which depend on the dimensions of the coil and are found from the curves shown in Fig. 4.

A study of Fig. 4 will show that the A factor accounts for the fact that coils of larger mean radii a establish larger ϕ's than do coils of smaller radii; and that the factor B accounts for the increasing *partial* linkage which accompanies an increase in the number of layers of the coil. For single-layer coils, both effects can be combined on a single curve. This single-curve method will be encountered in radio courses.

Example. Consider the 12-turn coil which is shown in cross-section in Fig. 4. a is the mean radius of the two-layer coil, and l is the axial length of the coil.

From the ratio a/l, the factor A in equation (21) is determined from the A curve as indicated in the sample calculation shown in Fig. 4.

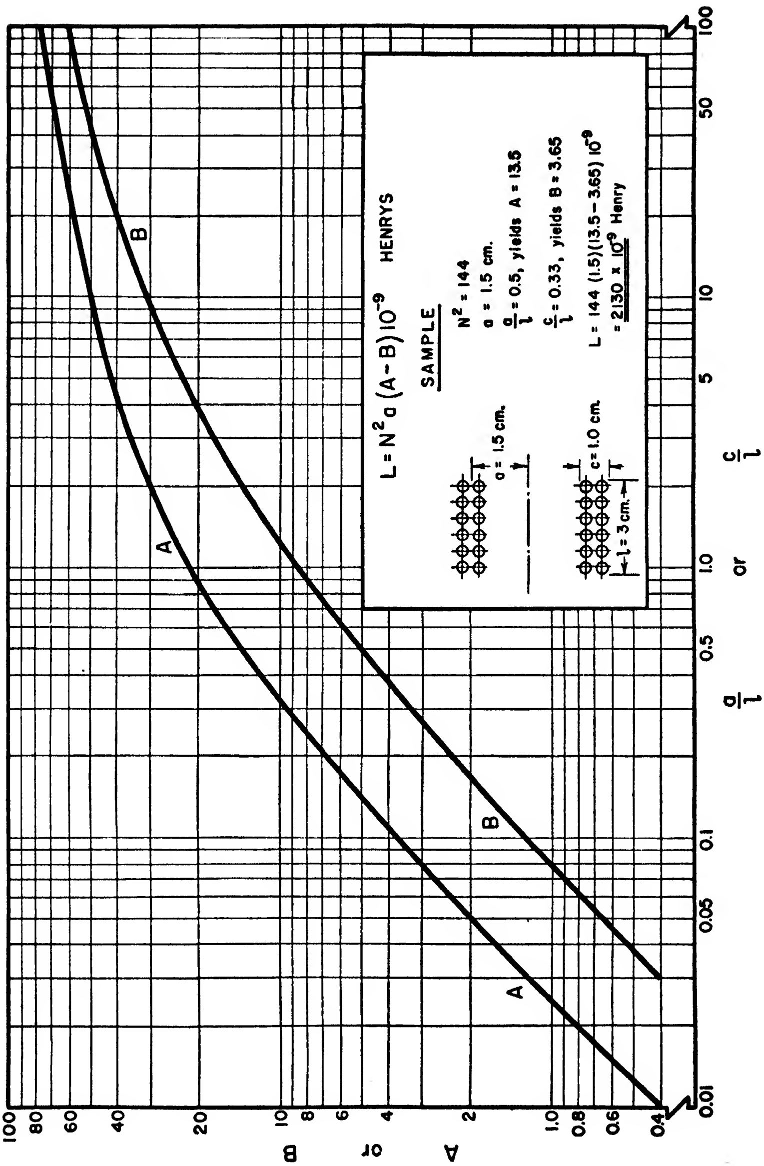


FIG. 4. Method for determining the self-inductance of straight solenoids.

From the ratio c/l, the factor B in equation (21) is determined from the B curve.

Substitution of the numerical values of N, a, A, and B into equation (21) then gives the self-inductance of the coil to an accuracy of about 2 per cent.

5. D-C and A-C Inductance of Iron-Cored Coils. In iron-cored coils, the magnetic flux is confined chiefly to the magnetic circuit and can be

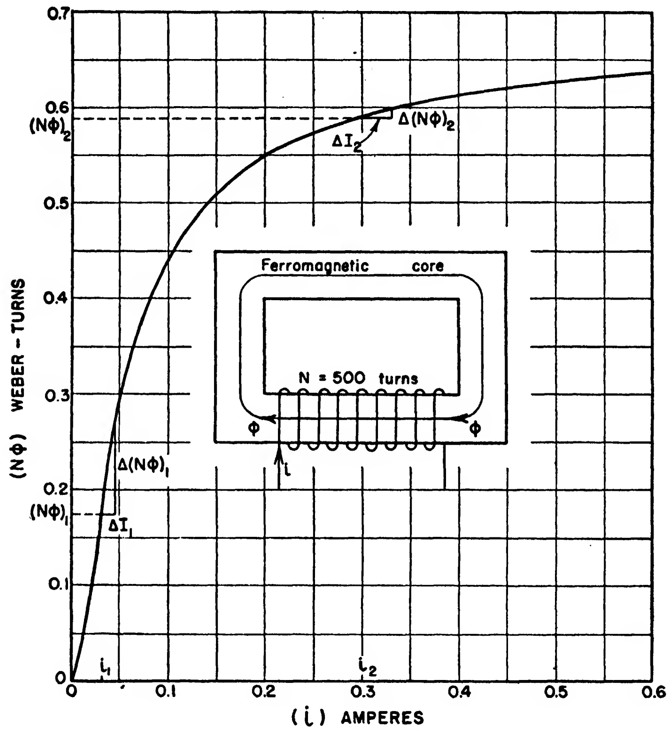


FIG. 5. The magnetization curve for a particular 500-turn iron-cored coil.

evaluated by methods which have been given in the preceding chapter. For a fixed core and a fixed number of turns, it is somewhat more convenient to use the magnetization curve ($N\phi$ versus I) which applies directly to the particular coil. A magnetization curve as shown in Fig. 5 is simply a $(B \times AN)$ versus $[H \times (l/N)]$ curve for fixed values of area A, number of turns N, and length of magnetic circuit l.

The exact self-inductance which is exhibited by an iron-cored coil like that shown in Fig. 5 will depend upon the magnitude of the magnetizing current i because upon this magnitude depends the number of flux linkages, and these flux linkages $N\phi$ are not linearly related to the

magnetizing current i. If, for example, the current is steady at the value $I = 0.2$ amp,

$$\text{(d-c inductance) } L_{0.2} = \frac{N\phi}{I} = \frac{0.55}{0.2} = 2.75 \quad \text{henrys}$$

but for $I = 0.05$ amp

$$\text{(d-c inductance) } L_{0.05} = \frac{N\phi}{I} = \frac{0.29}{0.05} = 5.8 \quad \text{henrys}$$

D-c inductances are of little importance in themselves since, under steady current conditions, $L(di/dt) = 0$, and the magnetic field does not react upon the electrical circuit. D-c inductances are, however, important in determining the *incremental* inductances which are considered in the following section.

If an alternating current (say, $i = I_m \sin \omega t$) magnetizes the core of Fig. 5, the instantaneous inductance is not constant at all times during the cycle of current due to the curvature of the hysteresis loop. If, however, the iron is *worked* not too far along on the upper bend of the magnetization curve, an approximate time-averaged value of a-c inductance throughout the working cycle is

$$L_{ac} = \frac{(N\phi)_m}{I_m} \quad \text{(optimistic value)} \tag{22}$$

where I_m is the maximum magnitude of the alternating current and $(N\phi)_m$ is the corresponding ordinate on the magnetization curve. The value of L_{ac} as given in equation (22) is somewhat higher than the actual value which is realized in practice owing to the demagnetizing action of the eddy currents in the laminations. (See Fig. 22, page 309.) It is plain that the magnitude of L_{ac} will depend upon the particular value of I_m which is present and will vary with the magnitude of I_m in the same manner as L_{dc} varies with the magnitude of the d-c exciting current.

The manner in which L_{ac} determines the maximum magnitude of the alternating current in a circuit like that shown in Fig. 5 when an a-c voltage is applied to the terminals of the coil will be considered presently.

6. Incremental and Differential Inductance. The magnetizing current in many iron-cored coils contains an a-c component superimposed on a d-c component as shown in Fig. 6. By way of illustration, we might consider the output transformer of an audio amplifier which necessarily carries a d-c component that is required to energize the output tube of the amplifier. Superimposed on this d-c component is the a-c signal component of current, and the latter component constitutes the useful

part of the total current because only a time-varying current can develop *changes* of flux in the iron core of the transformer. The changes of flux produced by I_{ac} are employed to generate $d\phi/dt$ voltages in the secondary winding of the transformer as, for example, a second winding which might be wound on the core shown in Fig. 5. The $d\phi/dt$ voltage generated in the secondary winding of the output transformer may be employed to drive the loudspeaker of the audio system.

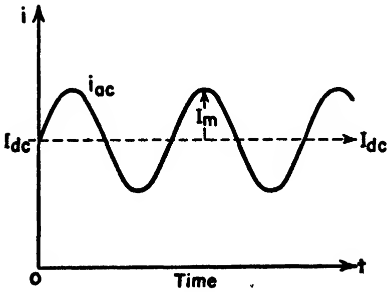


Fig. 6. Combined alternating and direct current making $i = I_{dc} + I_m \sin(2\pi ft)$.

In most instances where the magnetizing current takes the form shown in Fig. 6, it is desirable to obtain maximum *change* of $N\phi$ for a specified amplitude of the a-c component. This requires that

$$\Delta L = \frac{\Delta(N\phi)}{\Delta I} \quad \text{(incremental inductance)} \tag{23}$$

be as large as possible under the operating conditions encountered. In equation (23), ΔI is the maximum magnitude of the a-c component in Fig. 6, and $\Delta(N\phi)$ is the corresponding change in flux linkages as determined from the magnetization curve.

Plainly, equation (23) represents the slope of the magnetization curve *near* the $i = I_{dc}$ point on this curve. The slope of the curve *at* the $i = I_{dc}$ point is called the differential inductance and may be symbolized as

$$L_{\text{dif.}} = \frac{d(N\phi)}{di} \quad \text{(differential inductance)} \tag{24}$$

but this refinement over equation (23) is clearly not warranted when we consider that the actual $N\phi$ versus i variation follows a small hysteresis loop the center of which is at the operating (or $i = I_{dc}$) point.

Example 1. Let the current i in Fig. 6 be

$$i = 0.03 + 0.015 \sin(2\pi ft) \quad \text{amp}$$

where f is the frequency of the a-c component in cycles per second.

If this form of current flows through the coil of Fig. 5

$$L_{dc} = \frac{(N\phi)_1}{i_1} = \frac{0.175}{0.03} = 5.83 \quad \text{henrys}$$

as determined at the $i = i_1 = 0.03$ amp point as shown in Fig. 5.

$$(\text{incrementally})\ \Delta L = \frac{\Delta(N\phi)_1}{\Delta I_1} = \frac{0.095}{0.015} = 6.33 \quad \text{henrys}$$

In this case the incremental inductance is larger than the d-c inductance and larger than any value of total a-c inductance that could be obtained if the exciting current had no d-c component. The reason is plain; the d-c component (0.03 amp) places the operating point, $i_1, (N\phi)_1$, at the place on the curve where the differential inductance is maximum, and ΔI_1 (0.015 amp) is not too large to make the incremental inductance significantly different from the maximum slope of the magnetization curve.

Example 2. Let the current i in Fig. 5 and in Fig. 6 be

$$i = 0.30 + 0.033 \sin(2\pi f t) \quad \text{amp}$$

In this case

$$L_{dc} = \frac{(N\phi)_2}{i_2} = \frac{0.59}{0.3} = 1.97 \quad \text{henrys}$$

and

$$\Delta L = \frac{\Delta(N\phi)_2}{\Delta I_2} = \frac{0.01}{0.033} = 0.3 \quad \text{henry}$$

as shown in Fig. 5.

A comparison of these results with those obtained in Example 1 will illustrate clearly the effect of *working* the iron too far up on the magnetization curve.

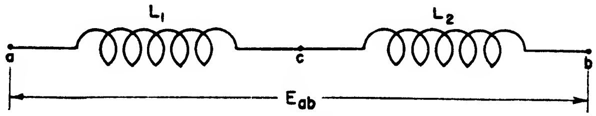


FIG. 7. Self-inductances in series. $L_{eq.} = L_1 + L_2$.

7. Series and Parallel Arrangements of Self-Inductances. Two self-inductances connected in series as shown in Fig. 7 combine to form an equivalent inductance of

$$L_{ab} = L_1 + L_2 \tag{25}$$

provided the resistances of the coils are neglected and provided none of the magnetic flux established by the L_1 coil links with the turns of the L_2 coil. Since the seriesed coils carry the same current I,

$$L_{ab} = \frac{(N_1\phi_1) + (N_2\phi_2)}{I} = L_1 + L_2$$

If the same coils are connected in parallel, L_{ab} is reduced in magnitude to a value which is lower than either L_1 or L_2 taken separately. In Fig.

8, it is plain that the voltage drops across the two coils are equal and in turn equal to the voltage across the terminals a and b. From equation (3),

$$\frac{di_1}{dt} = \frac{e_{ab}}{L_1} \qquad \frac{di_2}{dt} = \frac{e_{ab}}{L_2} \qquad \frac{di}{dt} = \frac{e_{ab}}{L_{ab}}$$

where e_{ab} is the time-varying voltage applied to terminals a and b of Fig. 8. Since $i = i_1 + i_2$, it may be readily shown that

$$L_{ab} = \frac{L_1 L_2}{L_1 + L_2} \quad \text{(for parallel connection)} \tag{26}$$

Where the resistive components of the voltage drops in the coils are significantly large or where the coils are coupled magnetically to one another, equations (25) and (26) cannot be used. Since these conditions can be handled much more easily in terms of the complex algebra em-

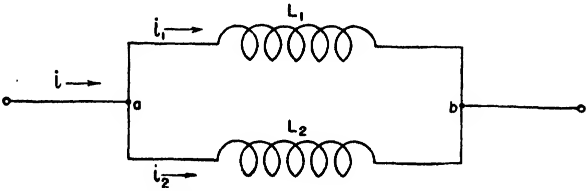


FIG. 8. Self inductances in parallel. $L_{eq.} = L_1L_2/(L_1 + L_2)$.

ployed in a-c circuits courses than they can be here, further consideration of series and parallel arrangements of inductances will not be given in this text.

8. Mutual Inductance and Coefficient of Magnetic Coupling. Two electric circuits are coupled magnetically when the magnetic field of one links with the turns of the other. The degree of coupling can be defined quantitatively in terms of the component fluxes shown in Fig. 9. In studying the component fluxes shown in Fig. 9 one recognizes that

$\phi_{11} + \phi_{12}$ is the total flux established by $N_1 i_1$.
$\phi_{22} + \phi_{21}$ is the total flux established by $N_2 i_2$.
ϕ_{11} is the flux which is established by $N_1 i_1$ not linking with N_2.
ϕ_{22} is the flux which is established by $N_2 i_2$ not linking with N_1.

If the reluctance of the mutual flux paths (the paths of ϕ_{12} and ϕ_{21}) is constant, the mutual inductance (from coil 1 to coil 2 or from coil 2 to coil 1) is defined as

$$M = \frac{N_2 \phi_{12}}{i_1} = \frac{N_1 \phi_{21}}{i_2} \tag{27}$$

where ϕ_{12} is the flux from N_1 which links with the N_2 turns
ϕ_{21} is the flux from N_2 which links with the N_1 turns.

Except where the turns of the two coils encircle the same flux path as in Fig. 10, page 290, it is usually impossible to calculate the mutual inductance which exists between two coils. Mutual inductance M can readily be measured in the laboratory, however. For the two coils shown in Fig. 9, we first connect terminal b of the N_1 coil to terminal c of the

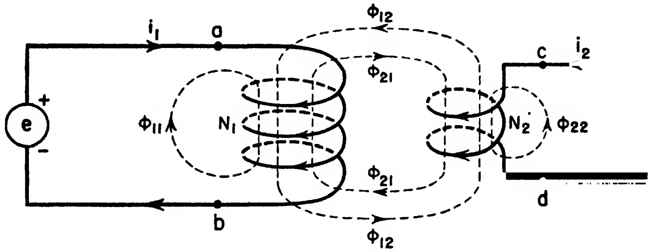


FIG. 9. Illustrating magnetic coupling between two electrical circuits.

N_2 coil to form a series circuit between terminals a and d, the inductance of which can be measured. In terms of component fluxes it can be seen that

$$L_{\text{sub.}} = \frac{N_1(\phi_{11} + \phi_{12} - \phi_{21})}{i} + \frac{N_2(\phi_{22} + \phi_{21} - \phi_{12})}{i}$$

(See Fig. 9 for the relative directions of ϕ_{12} and ϕ_{21}.) Since with this connection $i_1 = i_2 = i$ in equation (27) and since we recognize $N_1(\phi_{11} + \phi_{12})/i$ as L_1 and $N_2(\phi_{22} + \phi_{21})/i$ as L_2,

$$L_{\text{sub.}} = L_1 + L_2 - 2M \tag{28}$$

After the value of $L_{\text{sub.}}$ is obtained experimentally, the circuit is rearranged by connecting terminal b of N_1 to terminal d of N_2, and upon measuring the resultant inductance of the series circuit between terminals a and c there is found

$$L_{\text{add.}} = \frac{N_1(\phi_{11} + \phi_{12} + \phi_{21})}{i} + \frac{N_2(\phi_{22} + \phi_{21} + \phi_{12})}{i}$$

or

$$L_{\text{add.}} = L_1 + L_2 + 2M \tag{29}$$

If equation (28) is subtracted from equation (29), we find

$$M = \tfrac{1}{4}(L_{\text{add.}} - L_{\text{sub.}}) \tag{30}$$

and since $L_{\text{add.}}$ and $L_{\text{sub.}}$ can be readily measured, M can be found for any configurations of the circuits and for any placement of the circuits

relative to one another. Plainly, the value of M is given in whatever units $L_{\text{add.}}$ and $L_{\text{sub.}}$ are measured.

The coefficient of magnetic coupling between two circuits which individually possess L_1 and L_2 units of self-inductance is defined as

$$K_M = \frac{M}{\sqrt{L_1 L_2}} \tag{31}$$

This numeric is a measure of how closely the two electrical circuits are coupled. In iron-core transformers, K_M may exceed 0.98, whereas in most radio applications of coupled circuits the coefficient of coupling K_M is usually made to be in the order of 0.01 or 0.001 for best results.

Example. Let it be required to find the coefficient of magnetic coupling between coil N_1 ($L_1 = 0.20$ henry) and coil N_2 ($L_2 = 0.10$ henry) in Fig. 9 if

With terminal b connected to terminal c, $L_{ad} = 0.20$ henry
With terminal b connected to terminal d, $L_{ac} = 0.40$ henry

In this case

$$L_{ad} = L_{\text{sub.}} = 0.20 \quad \text{henry}$$

$$L_{ac} = L_{\text{add.}} = 0.40 \quad \text{henry}$$

$$M = 0.25(L_{\text{add.}} - L_{\text{sub.}}) = 0.25(0.40 - 0.20) = 0.050 \quad \text{henry}$$

$$K_M = \frac{M}{\sqrt{L_1 L_2}} = \frac{0.050}{\sqrt{0.2 \times 0.1}} = 0.354$$

9. Self-Inductance Analogous to Mass. In Section 1 of this chapter, the product Li in the electromagnetic system was compared to momentum mv in a mechanical system. This analogy may be carried forward to include all the electrical circuit quantities, and by so doing we may obtain a better intuitive grasp of the concept of self-inductance. For the simple systems shown in Fig. 10:

In the mechanical system		*In the electrical system*
f = driving force	is analogous to	e = driving voltage
$m\dfrac{dv}{dt} = f_m$ = inertial force	"	$L\dfrac{di}{dt} = v_L$ = inductive voltage
$K_d v = f_d$ = frictional force	"	$Ri = v_R$ = resistive voltage
$vt = l$ = displacement	"	$it = Q$ = electrical charge

where v is velocity, m is mass, t is time, l is length or displacement, and K_d is the damping constant which depends for its value upon the friction offered to the moving mass in Fig. 10-a.

Assuming that we close switch S in Fig. 10-b at some instant of time, say $t = 0$, we may write Kirchhoff's voltage equation for the simple circuit as

$$L\frac{di}{dt} + Ri = e \tag{32}$$

This equation says simply that the countervoltages $L(di/dt)$ and Ri must at all times balance the applied voltage e in order that dynamic equilibrium be maintained between the various voltages which are present in the system.

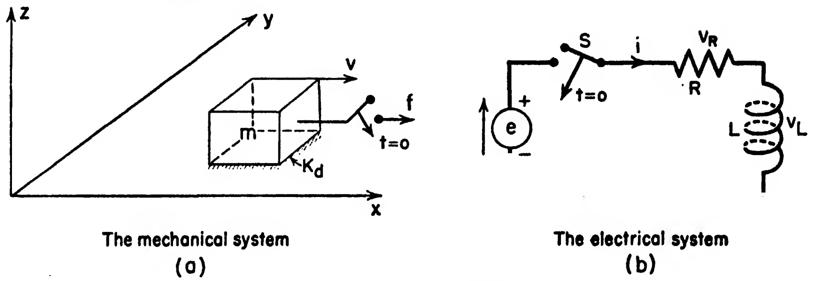


FIG. 10. An analogy.

In a similar manner, we know that for dynamic equilibrium of all the forces acting on the moving body the counter or restraining forces must equal the applied force which we shall imagine to be suddenly applied at $t = 0$. Thus

$$m\frac{dv}{dt} + K_d v = f \tag{33}$$

where the damping factor K_d is measured in force per unit velocity. In equation (33) we recognize $m(dv/dt)$ as the inertial (or ma) restraining force and $K_d v$ as the frictional force, the only two counter forces acting on the mass in Fig. 10-a.

In the mechanical system we can readily visualize the result of the sudden application of a force of constant magnitude for a particular length of time, say Δt beginning at $t = 0$. The inertial force is the only restraining force at the instant the force is applied if the velocity v is zero at this instant. The velocity of mass m will increase gradually as time t increases until finally the mass is moving at $v = F/K_d$ (where F is the magnitude of the applied force) provided our Δt is a sufficiently long period of time. After the driving force F is removed, the momentum mv will continue to move the mass forward until the kinetic energy possessed by mass m is dissipated in frictionally developed heat energy.

Since the governing differential equations of the two systems are

identical in form, we may expect that the time variation of current in Fig. 10-b will correspond precisely to the time variation of velocity in the mechanical system shown in Fig. 10-a. We shall solve the basic electrical equation given in (32) with a view toward showing that self-inductance L is responsible for a time retardation of the current response (to an applied voltage) which is analogous in all respects to the retardation of the velocity response of mass m to an applied force.

10. Charge in Motion Tends to Remain in Motion. A steady current, $I = E/R$, is flowing in the electrical circuit shown in Fig. 11. When the

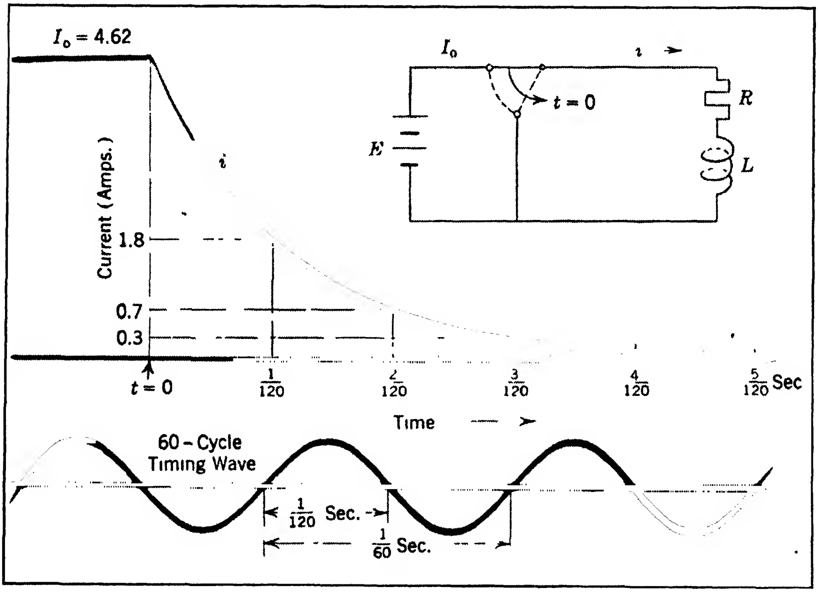


FIG. 11. Subsidence current in an LR series circuit. $I_0 = 30/6.5 = 4.62$ amp; $R = 6.5$ ohms; $L = 0.057$ henry.

driving voltage E is disconnected from the LR branch, this branch is closed on itself by the switching operation indicated in Fig. 11. After the battery voltage is removed the driving voltage of the closed loop is reduced to zero, and equation (32) becomes

$$L\frac{di}{dt} + Ri = 0 \qquad t \geqq 0 \tag{34}$$

subject to the initial condition that $i = I_0$ at $t = 0$.

In this simple case, the variables i and t may be separated and the equation solved for i by conventional methods of integration. But it is

evident that the solution for i must be of such form that, when added to the time derivative of this form, the sum will equal zero. In any case it will do no harm to assume a solution (as we often do in evaluating integrals) and test the solution to see if it satisfies equation (34). This method of solving differential equations is basically sound and is economical from a standpoint of time and effort wherever we have an intuitive understanding of the physical problem.

Let it be assumed that a solution to equation (34) takes the form

$$i = A\varepsilon^{\alpha t} \tag{35}$$

where A and α are constants yet to be determined and ε is the base of the natural logarithms. To test this solution we substitute in equation (34) the value of i which has been assumed with the following result

$$L(A\alpha\varepsilon^{\alpha t}) + RA\varepsilon^{\alpha t} = 0 \quad \text{or} \quad A\varepsilon^{\alpha t}(L\alpha + R) = 0$$

Thus the assumed form of i will satisfy equation (34) at all time provided

$$(L\alpha + R) = 0 \quad \text{or} \quad \alpha = -\frac{R}{L} \tag{36}$$

To evaluate the constant A we impose the initial condition ($i = I_0$ at $t = 0$) upon equation (35) and find, since $\varepsilon^0 = 1$, that

$$I_0 = A$$

Hence

$$i = I_0\varepsilon^{-Rt/L} \tag{37}$$

is the required solution of equation (34).

A graphical solution to equation (34) as obtained with an oscillograph is shown in Fig. 11. In certain cases, particularly where the circuit parameters R and L are variable, the oscillographic solution is the only feasible type of solution that can be obtained. It will be observed from Fig. 11 that, once current is established in an LR circuit in a given direction, it continues to flow in this direction until the energy stored in the magnetic field is returned to the electrical circuit where it is dissipated in the form of heat energy in the resistance R.

Had we suddenly opened switch S in Fig. 11 without closing the LR branch on itself, the energy stored in the magnetic field would be largely dissipated in the form of an arc at the blades of the switch. If the arc at the switch blades is ruptured too rapidly, the high values of $L(di/dt)$ voltages which are developed are likely to puncture the insulation of the inductance coil and produce additional arcing at this point.

Example. Let it be required to compare the graphical solution [to equation (34)] which is shown in Fig. 11 with that obtained directly from the analytical solution given in equation (37) at time $t = 1/60$ sec.

(1) From the oscillogram, we find $i = 0.7$ amp at $t = 1/60$ sec.

(2) Noting that $I_0 = 4.62$ amp and that $R/L = 6.5/0.057$ in this particular case, we have, from equation (37),

$$i = 4.62\varepsilon^{-114/60} = \frac{4.62}{\varepsilon^{1.9}} = \frac{4.62}{6.7} = 0.69 \quad \text{amp}$$

The correspondence between the solutions is closer than the accuracy with which the oscillogram can be read.

It might be inferred from the form of equation (37) that the current attains zero value only after an infinite period of time. The current actually comes to zero in a reasonably short period of time because the component voltage, which is responsible for the continued current flow, $L(di/dt)$, becomes so small that it cannot maintain a net drift of electrons against the internal atomic forces which tend to prevent net drifts of electrons within the metal conductors. Failure of equation (37) to account for this minute effect is of no practical importance or significance.

In illustrating that charge in motion I tends to remain in motion, we have shown that the magnetic field (with which the kinetic energy of the moving charge is associated) possesses a momentun which is defined quantitatively as $Li = N\phi$. Dimensionally this electromagnetic momentum may be thought of as either volts $\times$ seconds (since $N\phi = E^1t^1$) or as henrys $\times$ (coulombs/second), which is analogous to mass $\times$ velocity.

11. Charge at Rest Tends to Remain at Rest. If the LR circuit is initially at rest as shown in Fig. 12 and a constant driving voltage E is applied at $t = 0$, equation (32) becomes

$$L\frac{di}{dt} + Ri = E \tag{38}$$

subject to the initial condition that $i = 0$ at $t = 0$.

In the present case, the solution of equation (38) will plainly include a steady-state term since we know physically that, after a sufficiently long period of time has elapsed after $t = 0$ in Fig. 12, $i = E/R$. We shall therefore assume that the solution for i as a function of time takes the form

$$i = \underbrace{\frac{E}{R}}_{\text{steady}} + \underbrace{A_1\varepsilon^{\alpha t}}_{\text{transient}} \tag{39}$$

and test this assumption by putting this form of i back into equation

(38). The result of this test is

$$L(A_1\alpha\varepsilon^{\alpha t}) + R\left(\frac{E}{R} + A_1\varepsilon^{\alpha t}\right) = E$$

or

$$A_1\varepsilon^{\alpha t}(L\alpha + R) = 0$$

which is true for all time t if $\alpha = -R/L$ as in Section 10.

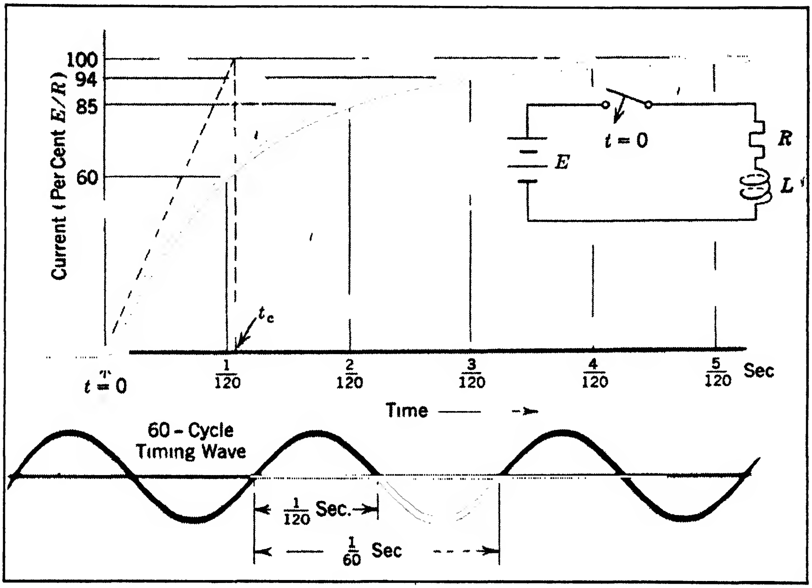


FIG. 12. Rise of current in an LR series circuit. $E = 30$ volts; $R = 6.5$ ohms; $L = 0.057$ henry.

Thus equation (39) is an acceptable solution of equation (38). Mathematicians call the steady-state term the *particular integral*, and the transient component the *complementary function*. Usually the particular integral is the more difficult to evaluate from a mathematical point of view but, since engineers usually know this portion of the solution, they follow the method employed here, namely that of writing into the general solution the known value of the steady-state term.

The constant A_1 may be evaluated by imposing the initial condition ($i = 0$ at $t = 0$) upon equation (39) to obtain

$$0 = \frac{E}{R} + A_1 \quad \text{or} \quad A_1 = -\frac{E}{R} \tag{40}$$

Hence

$$i = \frac{E}{R}(1 - \varepsilon^{-Rt/L}) \tag{41}$$

is the required solution for i as a function of time.

A plot of equation (41) for a particular set of values of E, L, and R is shown in Fig. 12. It will be observed that the current increases exponentially from zero to its E/R value and that it makes somewhat more than 60 per cent of this total transition in the period of time labeled t_c in Fig. 12.

Time Constant. The time rate at which current changes in the LR circuit is governed by the ratio L/R which is dimensionally equivalent to time since

$$L = \frac{e}{\dfrac{\Delta i}{\Delta t}} \qquad R = \frac{e}{\Delta i} \qquad \frac{L}{R} = \Delta t$$

Thus at $t = L/R$ in equation (41)

$$i = \frac{E}{R}(1 - \varepsilon^{-1}) = \frac{E}{R}\left(1 - \frac{1}{2.718}\right) = 0.632\,\frac{E}{R} \tag{42}$$

which says that, in the LR circuit, the current makes 63.2 per cent of its total change in time $t = t_c = L/R$. As applied to Fig. 12, $t_c = 0.057/6.5 = 0.00877$ sec.

In general, the time constant t_c of a circuit is defined as the length of time required for the current to make 63.2 per cent of the total change which occurs when the driving voltage is suddenly changed, that is, when the circuit is shock excited.

Example. Let it be required to find the length of time for the current in Fig. 12 to reach 0.94 (E/R). Equation (41) is to be employed.

Under the operating condition shown in Fig. 12,

$$\frac{E}{R} = 4.62 \quad \text{amp} \qquad \frac{R}{L} = 114 \quad \text{inverse sec.}$$

Equation (41) may be arranged as

$$4.62\varepsilon^{-114t} = 4.62 - (0.94)(4.62)$$

or

$$\varepsilon^{114t} = \frac{1}{0.06} = 16.67$$

Taking logarithms,

$$114t = \ln 16.67$$

and

$$t = \frac{2.81}{114} \doteq 0.0246 \quad \text{sec} \quad (\text{for } i = 0.94E/R)$$

As shown in the oscillogram, $i = 0.94(E/R)$ at about 3/120 or 0.025 sec.

During the period in which the circuit current changes, the magnetic flux linkages change, but a *change of flux linkage can be accomplished only in a finite period of time*. This guiding principle demands that $(Li)_0 = (N\phi)_0$ at $t = 0$ must equal $(Li)_{0+} = (N\phi)_{0+}$ *just after* any switching operation is performed. The symbol $0+$ means at the positive side of $t = 0$, but not a finite length of time after $t = 0$. In Fig. 11,

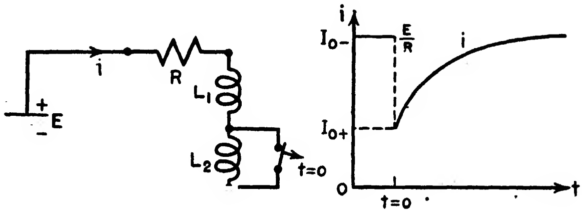


FIG. 13. Illustrating the principle of the conservation of LI.

$i = I_0$ at $t = 0$; therefore $i = I_0$ at $t = 0+$ since L is constant. In Fig. 12, $i = 0$ at $t = 0$; therefore, for the same reason, $i = 0$ at $t = 0+$.

12. Conservation of $LI = N\phi$ Comparable to Conservation of Momentum. Situations arise where the current through an inductance coil is caused to change abruptly. A case of this kind is illustrated in Fig. 13 where, before $t = 0$, a steady current $I = E/R$ is flowing through the L_1R circuit and zero current is flowing through L_2 owing to the short-circuited path around L_2.

At $t = 0$, the switch in Fig. 13 (presumably in a vacuum where arcing will not occur) is opened. Just prior to $t = 0$, L_1 carries E/R current and L_2 carries zero current. Just after $t = 0$, $I_{L_1} = I_{L_2}$ as demanded by the series connection and the assumption that no arcing occurs at the blades of the switch.

Kirchhoff's voltage equation for the circuit shown in Fig. 13 during the period of time starting at $t = 0$ is

$$(L_1 + L_2)\frac{di}{dt} + Ri = E \tag{43}$$

subject to the condition that Li before the switching operation equals Li after the switching operation.

The solution of equation (43) is of the same form as that given in equation (39) for the solution of equation (38), since (43) and (38) are of the same form. Thus

$$i = \frac{E}{R} + A_2\varepsilon^{\alpha t} \tag{44}$$

where, in this case, it is readily shown that $\alpha = -R/(L_1 + L_2)$ or, by letting the sum of L_1 and L_2 be known as L_s,

$$\alpha = -\frac{R}{L_s} \tag{45}$$

In evaluating A_2 in this case, we must rely upon the principle of the conservation of Li (or of $N\phi$) to guide us because we do not know from inspection just what value i in equation (44) has at $t = 0+$.

Applying the conservation of Li principle,

$$Li \text{ (before the switching operation)} = L_1 \frac{E}{R}$$

$$Li \text{ (just after switching operation)} = (L_1 + L_2)I_{0+}$$

Since the above values of Li must be equal, we find

$$I_{0+} = \frac{L_1}{L_1 + L_2} \times \frac{E}{R} \quad (\text{at } t = 0+) \tag{46}$$

which if imposed on equation (44) yields

$$\frac{L_1}{L_1 + L_2} \times \frac{E}{R} = \frac{E}{R} + A_2$$

or

$$A_2 = \frac{E}{R}\left(\frac{-L_2}{L_1 + L_2}\right) = \frac{E}{R}\,\frac{-L_2}{L_s} \tag{47}$$

Hence

$$i = \frac{E}{R}\left(1 - \frac{L_2}{L_s}\varepsilon^{-Rt/L_s}\right) \tag{48}$$

where $L_s = L_1 + L_2$.

Example. Let it be required to find

(1) I_{0+} in Fig. 13 when the switch is opened.

(2) The length of time required for i to make 63.2 per cent of its transition back to the E/R value after the sudden decrease caused by the switching operation.

The known data are: $E = 30$ volts; $R = 5$ ohms; $L_1 = 0.2$ and $L_2 = 0.3$ henry.

$$\text{Just before } t = 0 \text{ (or at } t = 0-\text{)}, \; I_{0-} = \frac{E}{R} = 6 \quad \text{amp}$$

$$\text{At } t = 0+, \; I_{0+} = \frac{L_1}{L_1 + L_2} \times \frac{E}{R} = \frac{0.2}{0.5} \times 6 = 2.4 \quad \text{amp}$$

Thus the current drops sharply in magnitude at $t = 0$ from 6 amp to 2.4 amp and will return to the 6-amp value (assuming that the L_2 coil has a negligibly small resistance) in a period of time which is governed by the time constant of the circuit after $t = 0$, namely, by L_s/R.

The time required for i to recover to $[0.632(6 - 2.4) + 2.4]$ or 4.68 amp

$$t_c = \frac{L_1 + L_2}{R} = \frac{0.5}{5.0} = 0.1 \quad \text{sec}$$

This recovery of i to 4.68 amp in 0.1 sec may be checked by means of equation (48).

In using the conservation of Li as a basis of calculation we are imposing the condition that flux linkages $N\phi$ cannot change instantly any more than the momentum of a moving mass can change instantly. Thus the principle of the conservation of Li is to the electromagnetic system what the principle of the conservation of momentum mv is to the mechanical system.

13. Determination of L from a Current-Time Graph. The instantaneous magnitude of L in an LR circuit may be evaluated at any point on the current-time graph of this circuit if the voltage applied is constant as it is in Figs. 12 and 14. From the basic voltage equation of the circuit

$$L_1 = \frac{E - Ri_1}{\left(\dfrac{di}{dt}\right)_1} \tag{49}$$

where the subscripts indicate that L, i, and di/dt are to be evaluated at the same point on the current-time graph.

Where μ_r of the flux paths is constant as it is in Fig. 12, L is constant and, except as it affects the accuracy of the graphical construction, the evaluation may be performed at any point on the current-time graph.

Where μ_r is variable as it is in Fig. 14, the value of L will be found to vary at different points along the current-time graph, as might be expected from the shape of the magnetization curve. (See Fig. 5.) In the oscillogram shown in Fig. 14, the abrupt rise in current at $t = 0$ is attributable to the same effect as is the lower knee of the magnetization curve, namely, the apparent inertia that the sub-atomic current loops

have to initial orientation. It will be observed from the oscillogram, however, that orientation is in full progress in considerably less than 0.001 sec.

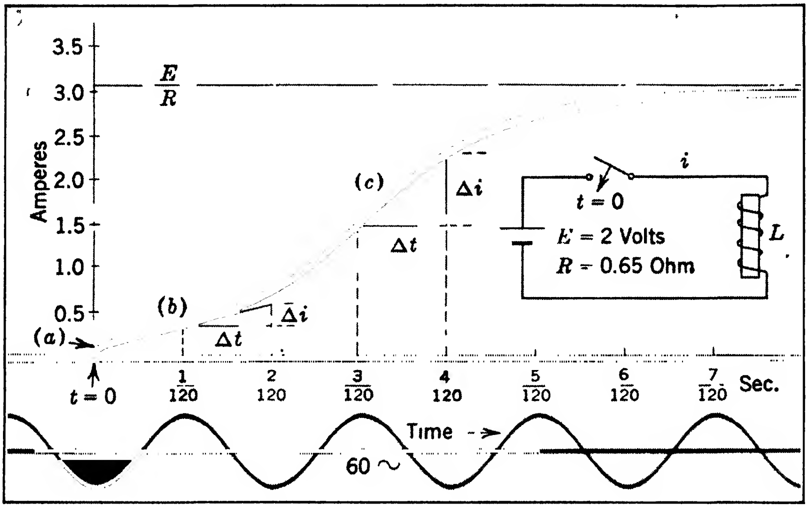


FIG. 14. Current-time graph; L is variable.

Example 1. Let it be required to determine L in Fig. 12 from the initial slope $(di/dt)_0$ of the current-time graph.

By graphical measurements,

$$\left(\frac{di}{dt}\right)_0 = \frac{\frac{E}{R}}{t_c} = \frac{4.62}{0.0088} = 525 \quad \text{amps/sec}$$

$$L = \frac{E - Ri_0}{\left(\frac{di}{dt}\right)_0} = \frac{30 - 0}{525} \doteq 0.057 \quad \text{henry}$$

Example 2. Let it be required to find the *time-average value* of inductance over the first 1/60-sec time interval of the current-time graph shown in Fig. 14.

In this case we draw a straight line through the mid-point of the time interval (at $t = 1/120$ sec in this case) which is most nearly representative of the average slope of the curve over the time interval specified, 1/60 sec. The upper part only of this straight line is shown in Fig. 14 since the average di/dt over the 1/60 sec time interval can as well be evaluated on a half-interval basis.

$$L_{\text{av.}} = \frac{E - Ri_b}{\left(\frac{\Delta i}{\Delta t}\right)_b} = \frac{2 - (0.65 \times 0.03)}{0.25 \times 120} = 0.06 \quad \text{henry}$$

14. Steady-State Current in the *LR* Circuit with A-C Driving Voltage. If a force were applied first in one direction and then in the other to mass m in Fig. 10-a, we can visualize the general nature of the velocity response of the mass to the alternating force. Owing to momentum, the velocity will continue in one direction after the force is operating in the reverse direction; and, as a result of the momentum, the velocity response will *lag in time* the alternations of the applied force. The maximum magnitude of the velocity will clearly depend upon the mass and the rapidity with which the force reversals occur. If the frequency of the force reversals is sufficiently high, it is conceivable that no perceptible velocity response will occur.

A corresponding situation is present in many radio circuits where frequencies of about 10^6 cycles/sec are effectively prevented from entering certain parts of the network by the simple expedient of placing a small choke coil in the paths where the radio-frequency signal is not wanted. These radio-frequency choke coils are often of 0.10 henry self-inductance and are represented by *rfc* on radio circuit diagrams.

It will be shown presently that the impedance offered by a pure inductance L to the flow of alternating current is

$$\text{inductive reactance } X_L = \omega L \quad \text{ohms} \tag{50}$$

where $\omega = 2\pi f$ is the *angular* frequency of the alternating current expressed in radians per second and f is the frequency in cycles per second. Using equation (50) to determine the inductive reactance of a 0.10-henry coil at a frequency of 10^6 cycles/sec, we find

$$X_L = \omega L = (6.28 \times 10^6) \times 0.1 = 628{,}000 \quad \text{ohms}$$

In order to appreciate the fact that ωL is an impedance and expressible in ohms, we may proceed as follows. Assume that a sinusoidally time-varying current is flowing through the inductance L. A time-varying current of sinusoidal wave form may be represented by the equation

$$i = I_m \sin(\omega t - \theta) \quad \text{amp} \tag{51}$$

where I_m is the maximum magnitude of the sinusoidally varying current
$\omega = 2\pi f$ is the angular frequency as previously defined
θ is, for the present, *any constant angle* we choose to select.

Under these conditions, the countervoltage of self-inductance [which is developed within the coil and which opposes the applied sinusoidal voltage that produces $i = I_m \sin(\omega t - \theta)$] is

$$v_L = L\frac{di}{dt} = \omega L I_m \cos(\omega t - \theta) = V_{Lm} \cos(\omega t - \theta) \quad \text{volts} \tag{52}$$

It will be observed that the time-varying v_L has a maximum value of $\omega L I_m$ which we shall call V_{Lm}. It is evident that

$$\frac{V_{Lm}}{I_m} = \omega L \quad \text{ohms} \tag{53}$$

and that, even though the inductance coil L possesses almost zero resistance, the maximum current I_m will never be greater than the maximum voltage applied to the LR circuit divided by ωL since V_{Lm} will never be greater than the maximum applied voltage in this type of circuit.

In order to understand the precise nature of the time lag of the current in the LR circuit relative to the a-c driving voltage, we need only add to v_L as given in equation (52) the Ri counter-voltage which is present in the circuit when $i = I_m \sin(\omega t - \theta)$ amp flow in the LR circuit. Thus

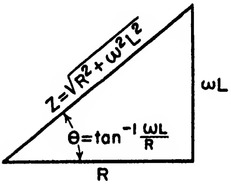


FIG. 15. For simplifying equation (54-a).

$$v_L + v_R = e \tag{54}$$

where v_L is given by equation (52), $v_R = RI_m \sin(\omega t - \theta)$, and e is the driving voltage which is applied to the LR circuit and which we can readily evaluate for the assumed current of $I_m \sin(\omega t - \theta)$. Substituting the known values of v_L and v_R into equation (54),

$$\omega L I_m \cos(\omega t - \theta) + RI_m \sin(\omega t - \theta) = e \tag{54-a}$$

To combine the two trigonometric terms on the left-hand side of equation (54-a) into a single term we may use the trigonometric substitution suggested in Fig. 15; that is,

$$\frac{\omega L}{Z} = \sin\theta \quad \text{and} \quad \frac{R}{Z} = \cos\theta$$

With the aid of these substitutions, the left-hand side of equation (54-a) reduces to a simple and recognizable form. Thus

$$ZI_m\,[\sin\theta\cos(\omega t - \theta) + \cos\theta\sin(\omega t - \theta)] = e \tag{54-b}$$

and combining the trigonometric terms as

$$(\sin\alpha\cos\beta + \cos\alpha\sin\beta) = \sin(\alpha + \beta)$$

we have

$$ZI_m \sin\omega t = e \quad [\text{applied voltage for } i = I_m \sin(\omega t - \theta)] \tag{55}$$

where $Z = \sqrt{R^2 + \omega^2 L^2}$ is the impedance of the LR circuit in ohms

$\theta = \tan^{-1}(\omega L/R)$ is the angle by which i lags e, the applied voltage.

Since the maximum magnitude of the applied voltage e is $E_m = ZI_m$,

$$i = \frac{E_m}{Z} \sin (\omega t - \theta) \quad \text{amp} \tag{56}$$

It is plain that the maximum magnitude of current that will flow in the LR circuit, I_m, for any specified maximum value of voltage E_m will depend upon the impedance Z. The impedance Z may in turn be essentially equal to ωL if the resistance of the circuit is small relative to ωL, and in this case the current *lags* the applied voltage in time by an angle which approaches 90°, or one-quarter cycle.

The time relationship between the applied voltage and the resulting current in a purely inductive circuit is shown in Fig. 16, and a study of

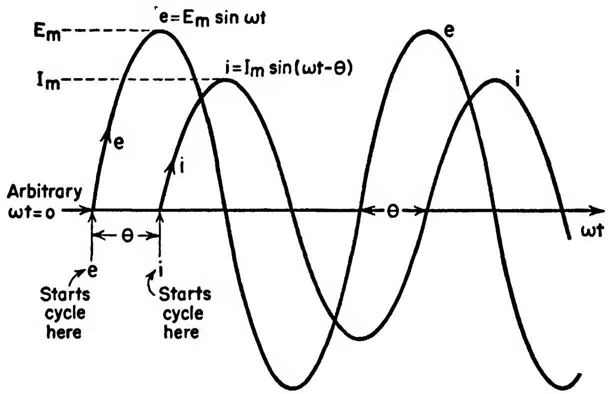


FIG. 16. Illustrating the time lag of the current variation relative to the applied a-c voltage. One complete cycle of current or voltage variation corresponds to 2π radians or 360°.

this figure will show what is meant by self-inductance causing the current to *lag* the applied voltage by a time angle of $\theta°$ or θ radians.

The time lag of current relative to the voltage for the case of a variable inductance is shown in Fig. 17. The non-sinusoidal wave form of the current is typical of all iron-cored coils which are energized with ordinary a-c voltages. The detailed analysis of the non-sinusoidal case will be left to more advanced courses.

Example. Assume that in Fig. 10-b the switch has been closed for a sufficiently long period so that all transient disturbances have died away and that

$$e = 200 \sin (377t) \text{ volts} \qquad R = 20 \text{ ohms} \qquad L = 0.10 \text{ henry}$$

For these values of R and L, the switch in Fig. 10-b needs to be closed for only about 0.05 sec for the transient component of the current to have died

out and for the steady-state current $i = I_m \sin(\omega t - \theta)$ to have established itself as the circuit current.

Let it be required to find (1) the maximum value of the steady-state current in amperes, (2) the angle of lag of the current i relative to the voltage e.

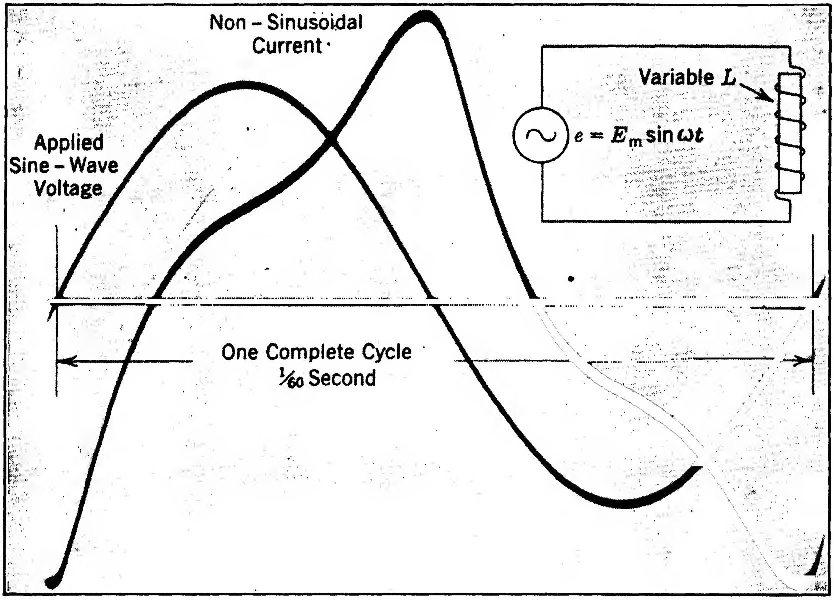


FIG. 17. Illustrating the time lag of a non-sinusoidal current relative to the applied voltage.

We first observe from the specified voltage that the angular frequency is 377 radians/sec ($f = 377/2\pi = 60$ cycles/sec).

$$\omega L = 377 \times 0.10 = 37.7 \text{ ohms} \quad \text{(of inductive reactance)}$$

$$R = 20 \text{ ohms} \quad \text{(specified)}$$

$$Z = \sqrt{R^2 + \omega^2 L^2} = \sqrt{20^2 + 37.7^2} \doteq 42.6 \text{ ohms} \quad \text{(impedance)}$$

$$I_m = \frac{E_m}{Z} = \frac{200}{42.6} = 4.7 \text{ amp} \quad \text{(required value of } i\text{)}$$

$$\theta = \tan^{-1}\frac{\omega L}{R} = \tan^{-1}\frac{37.7}{20.0} \doteq 62° \quad \text{(required angle of lag)}$$

With this information we may write the expression for the instantaneous current as

$$i = I_m \sin(\omega t - \theta) = 4.7 \sin(377t - 62°) \quad \text{amp}$$

where it is understood that, if we should ever combine $377t$ radians and the

62°, one or the other will have to be transformed. The 62° here means that i lags e by slightly more than ⅙ of a cycle.

15. Summary. (1) Self-inductance is symbolized by L and is defined as

$$L = \frac{N\phi}{I} \qquad \frac{\text{weber-turns}}{\text{amp}} \quad \text{(called henrys)} \tag{57}$$

(2) The energy stored in the magnetic field which links with a current-carrying circuit is accounted for in terms of the circuit parameter L:

$$\text{stored energy} = \frac{Li^2}{2} \quad \text{joules} \tag{58}$$

(3) The numerical value of L depends only upon the physical dimensions of the turns of the circuit, the number of turns, and the permeability of the linkage-flux paths, and can be determined by calculations of $(N\phi)/I$ in closed solenoids, parallel-wire lines, and in coaxial cable.

(4) The self-inductance of coils may be found from empirical formulas, one of which is given in Fig. 4, page 325.

(5) The mutual inductance between two circuits is defined as

$$M = \frac{N_2\phi_{12}}{i_1} = \frac{N_1\phi_{21}}{i_2} \quad \text{(See Fig. 9.)} \tag{59}$$

and the coefficient of coupling between the two circuits is defined as

$$K_M = \frac{M}{\sqrt{L_1L_2}} \tag{60}$$

(6) The basic voltage equation of the LR series circuit is

$$L\frac{di}{dt} + Ri = e \tag{61}$$

(7) If the driving voltage e in equation (61) is constant at E volts and $i = 0$ at $t = 0$, the current-time relationship is

$$i = \frac{E}{R}(1 - \varepsilon^{-Rt/L}) \quad \text{amp} \tag{62}$$

(8) If the driving voltage e in equation (61) is $e = E_m \sin \omega t$, the steady-state alternating current which flows in the LR series circuit is

$$i = \frac{E_m}{Z}\sin(\omega t - \theta) \quad \text{amp} \tag{63}$$

The meanings of E_m, Z, ω, and θ are given in the preceding section.

(9) Two methods of experimentally determining L are

(*a*)

$$L_1 = \frac{E - Ri_1}{\left(\frac{di}{dt}\right)_1} \quad \text{from current-time graphs like Figs. 11, 12, and 14}$$

(*b*)

$$L = \frac{\sqrt{Z^2 - R^2}}{\omega} \quad \text{under sinusoidal driving voltages where } Z = \frac{E_m}{I_m}$$

A-c bridge methods which are more accurate than either (*a*) or (*b*) are usually employed in actual practice, but the theory of a-c bridges is not given here.

PROBLEMS

1. Assume that the actual $N\phi$ versus i characteristic of a particular iron-cored coil which is carrying less than 1 amp current can be approximated as the straight line between $i = 0$ and $i = 1$ amp in Fig. 18.

(*a*) What is the value of L in this region of operation?

(*b*) What energy is stored in the magnetic field at $i = 0.5$ amp?

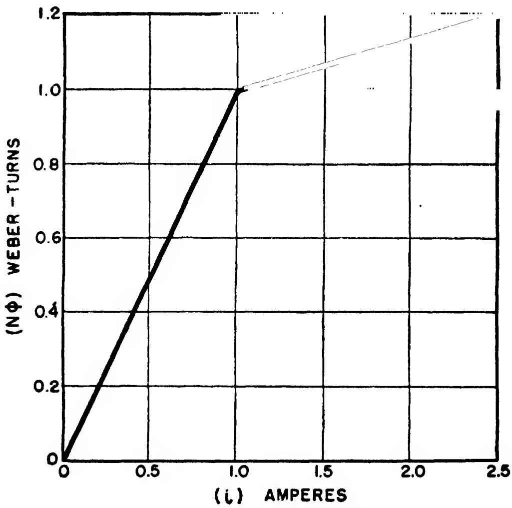


FIG. 18. For use in connection with Probs. 1, 13, and 14.

2. A current of 20 amp is flowing through an inductance of 2.8 henrys when the current is suddenly broken (say by a circuit breaker) in 0.05 sec.

(*a*) What is the time-averaged voltage of self-inductance generated by the collapsing magnetic field?

(*b*) What is the time-averaged power (in watts) of the energy dissipated in the circuit resistance and at the breaker arc during the 0.05-sec period?

3. A ring-shaped core like that shown in Fig.1 , page 319, is wound with a single layer winding of 2738 turns. Find L if $\mu_r = 1$, $r_1 = 25$ cm, $r_2 = 35$ cm, and $b = 20$ cm.

4. The core of Fig. 1, wound with 4000 turns of No. 30 B. & S. gage wire, has been shown to have a self-inductance of 0.029168 henry where

$$\mu_r = 1 \quad r_1 = 20 \text{ cm} \quad a = 4 \text{ cm} \quad b = 5 \text{ cm}$$

The calculation of L in the example of Section 2 was based on the assumption that the flux area was confined to just the core area.

What is the self-inductance based on the refinement that the flux area actually extends from center-of-wire to center-of-wire, that is,

$$\text{flux area} = (a + d)(b + d)$$

where d is the diameter of the wire? No allowance is to be made for insulation or for bulging of the turns.

5. The radius of the inner conductor of a coaxial (copper) cable is 0.02 in., and the inner radius of the outer conductor is 0.20 in. $\mu_r = 1$ throughout the region of the linkage-flux paths. Find the self-inductance of 1 km of this cable at ultra-high-frequencies where $L_{\text{int.}}$ of both conductors is essentially zero. Express the result in millihenrys.

6. What is the self-inductance of a 100-mile length of parallel-wire circuit which is composed of No. 14 B. & S. gage copper wires separated center-to-center in air by a distance of 24 in.?

7. An unorthodox transmission system consists of two cylindrical parallel conductors (a and b) spaced 80 in. center-to-center in air. Conductor a is a copper wire having a diameter of 0.10 in. Conductor b is an iron wire having a diameter of 1.0 in. and a space-averaged relative permeability of 100. Assuming uniform current density in both conductors, find the loop inductance (both wires) in henrys per meter.

8. If the coil shown in cross-section in Fig. 4, page 325, is extended axially by winding on an additional 12 turns of wire, what is the self-inductance of the 2-layer 24-turn coil, and why is it less than $24^2/12^2$ times the inductance of the 12-turn coil, namely, 2.13 microhenrys?

9. If the coil shown in cross-section in Fig. 4, page 325, is extended radially by winding an additional 12 turns of wire over the outside of the two layers shown in Fig. 4, what is the self-inductance of the 4-layer, 24-turn coil?

10. A cylindrical 1-layer coil of 20 tightly packed turns of wire has a mean radius (a in Fig. 4) of 1 in. and an axial length of 2 in. What is the self-inductance of the coil expressed in microhenrys?

11. (*a*) What is the d-c inductance in Fig. 5, page 326, at $i = 0.5$ amp?
(*b*) What is the incremental (or differential) inductance at $i = 0.5$ amp?

12. A magnetizing current $i = (0.5 + 0.05 \sin 377t)$ amp flows in the 500-turn coil shown in Fig. 5, page 326, and it is assumed that a second coil is wound directly over the 500-turn magnetizing coil but not connected electrically to the magnetizing coil.

If the second coil has 1000 turns, what is the maximum magnitude of the magnetically induced voltage in this coil by the small time-varying component of the magnetizing current?

13. What is the incremental inductance of the iron-cored coil whose magnetization curve is shown in Fig. 18: (*a*) in the operating region between $i = 0$ and $i = 1.0$ amp; (*b*) in the operating region between $i = 1$ and $i = 2.5$ amp?

14. A magnetizing current $i = (0.5 + 0.1 \sin \omega t)$ amp flows in the iron-cored coil whose magnetization curve is shown in Fig. 18.

(*a*) What is the maximum magnitude of the $d(N\phi)/dt = L(di/dt)$ voltage generated in this winding if $\omega = 5000$ radians/sec?

(*b*) What is the maximum magnitude of the voltage generated in a second winding which encircles the same iron core as does the magnetizing winding if this second winding has 3 times as many turns as the magnetizing winding?

(*c*) What is the frequency of the voltage generated in the second winding expressed in cycles per second?

15. A coil L_1 of 1000 turns establishes a total flux of 10^{-4} weber/amp of exciting current, 60 per cent of which links with a 500-turn coil L_2. The permeability of the flux paths is constant and, as a result, L_2 establishes 0.5×10^{-4} weber/amp of exciting current, 60 per cent of which links with the 1000-turn coil.

(*a*) What is the mutual inductance of the two coils?

(*b*) What is the coefficient of magnetic coupling between the coils?

16. Given two inductance coils which individually possess self-inductances of $L_1 = 16$ and $L_2 = 4.0$ millihenrys. The coils are in fixed space positions relative to one another and in these positions have a magnetic coupling coefficient of 0.30.

(*a*) What value of two-terminal self-inductance is obtained by connecting these coils in subtractive series?

(*b*) Using these two coils, what maximum two-terminal self-inductance can be obtained?

17. What minimum two-terminal self-inductance can be obtained using the two coils, L_1 and L_2, described in Prob. 16 individually, in series, *or* in parallel. *Note:* In the subtractive parallel connection

$$L_1 \frac{di_1}{dt} - M \frac{di_2}{dt} = e \quad \text{(applied to parallel branches)}$$

$$L_2 \frac{di_2}{dt} - M \frac{di_1}{dt} = e \quad \text{`` `` `` ``}$$

$$L_{eq.} \left(\frac{di_1}{dt} + \frac{di_2}{dt} \right) = e \quad \text{`` `` `` ``}$$

In solving for $L_{eq.}$, the time derivatives may be treated as ordinary algebraic variables like i_1 and i_2.

18. Two inductance coils connected in additive series show a measured two-terminal value of inductance of 56 millihenrys; and in subtractive series show a measured value of 14 millihenrys. What is the mutual inductance between the two coils?

19. A current of 10 amp is flowing in an LR series circuit when, at $t = 0$, the driving voltage is removed and the LR series branch short-circuited on itself. $R = 4$ ohms and $L = 1.0$ henry.

(*a*) What length of time is required for the current to subside to 0.1 amp?
(*b*) What energy remains stored in the magnetic field at $t = 0.50$ sec?

20. From the data given in connection with Fig. 11, page 334, calculate the current at $t = 1/120$ sec by $i = I_0\epsilon^{-Rt/L}$ and compare the result obtained with the measured value of 1.8 amp shown in Fig. 11.

21. From the data given in connection with Fig. 12, page 337, calculate the current at $t = 1/120$ sec and at $t = 1/60$ sec by $i = (E/R)(1 - \epsilon^{-Rt/L})$ and compare the values thus found with those shown on the current-time graph of Fig. 12.

22. What is the time constant of an iron-cored coil which has 10 ohms resistance and an inductance which is assumed to be constant at 30 henrys?

23. What is the instantaneous value of self-inductance in Fig. 14, page 342, at $t = 3/120$ sec?

24. In Fig. 13, page 339, $E = 30$ volts; $R = 5$ ohms; $L_1 = 0.20$ henry; $L_2 = 0.30$ henry. As indicated the switch is opened at $t = 0$.
(*a*) What is the ampere value of I_{0+}?
(*b*) What is the value of the circuit current 0.1 sec after the switch is opened?
(*c*) What is the final value of the circuit current i after the switch is opened?

25. What is the inductive reactance of a 0.15-henry inductance coil at a frequency of 796 cycles/sec?

26. If a 0.15-henry coil has a resistance of 100 ohms, what is the impedance of this coil at $\omega = 5000$ radians/sec?

27. A sinusoidal driving voltage, $e = 300 \sin (5000\,t)$ volts, is applied to the terminals of the coil described in Prob. 26. (In e, t is in seconds.)
(*a*) By what fraction of a cycle does the circuit current lag the applied voltage?
(*b*) What is the maximum magnitude of the circuit current?
(*c*) Write the expression for the instantaneous current as

$$i = I_m \sin (\omega t - \theta)$$

where I_m, ω, and θ are all expressed in numerical values.

28. Refer to Fig. 10-a, page 333. At $t = 0$, a constant and sustained force of 20 newtons is applied to the mass, $m = 30$ kg. It is assumed that the damping factor K_d is constant at 2 newtons/m/sec.
(*a*) What is the final steady-state velocity of the mass?
(*b*) What length of time is required for the velocity to reach 63.2 per cent of the final steady-state velocity?
(*c*) What is the kinetic energy of the mass 30 sec after the force is applied?

29. A sinusoidal time-varying force, $f = 4.5 \sin (6.28t)$ lb, is applied to the mass of Fig. 10-a where it is assumed that the damping factor K_d is constant at 0.137 lb/ft/sec. The mass is a *weight* of 66.16 lb or a 2.055-slug mass. In the expression for force, t is expressed in seconds, and it is assumed that this force has been applied for some time so that the velocity has reached a steady-state sinusoidal variation.
(*a*) What is the frequency of the alternating force in cycles per second?

(*b*) What maximum velocity is attained by the mass during a cycle?

(*c*) What is the time angle of lag of the time-varying velocity relative to the sinusoidally varying force?

30. Without regard for direction, what is the total linear distance traversed by the mass in Prob. 29 in 10 complete cycles, that is, in a period of 10 sec?

Note: Problems 28, 29, and 30 may all be worked in terms of the analogous electrical equations or the necessary relationships may be derived from equation (33) in precisely the same manner as the electrical equations were derived from equation (32).

CHAPTER XIII

Capacitance

1. Concept and Definition of Capacitance *C*. It has been shown in Chapter VII that energy stored in an *electric* field is electric potential energy. In general, this stored potential energy comes into being wherever positive charge is separated from negative charge; and, in particular, wherever two conductors are maintained at a potential difference relative to one another.

If two conductors which are at an electric potential difference are parts of an electric circuit, the stored potential energy will react on the circuit, releasing energy to the circuit when the potential difference is lowered and demanding increased energy from the driving source when the potential difference is increased. The interaction between an electric field and an electric circuit is accounted for in terms of a circuit parameter called *capacitance* which is defined as

$$C = \frac{Q}{V_c} \qquad \frac{\text{coulombs}}{\text{volt}} \text{ or farads} \quad \text{(in mks units)} \tag{1}$$

Specifically, this definition says that the capacitance of two conductors (one charged with $+Q$ coulombs and the other with $-Q$ coulombs) is the ratio of *one* of these charges and the potential difference between the conductors expressed in volts. This potential difference is that which is established between the conductors as a result of the separation of charge ($+Q$ on one conductor and $-Q$ on the other), the assumption being that the conductors are electrically neutral before the separation of charge is effected.

As applied to Fig. 1, it is plain that the two parallel plates will charge up to a point where V_c is a countervoltage which, when equal in magnitude to E, will reduce the current in the connecting wires to zero. If the magnitude of E is increased, positive current flows to the right as shown in Fig. 1 until the increase in $Q/C = V_c$ again balances E, thereby reducing the current to zero. If the magnitude of E is decreased positive current flows to the left as shown in Fig. 1 because, with $Q/C = V_c$ greater than E, charge is released from the parallel plates until $V_c = E$.

Applying Kirchhoff's voltage law to the arrangement shown in Fig. 1, we have

$$V_c = \frac{Q}{C} = E \quad \text{(voltage equation)} \tag{2}$$

If the above equation is multiplied by current ($i = dq/dt$), there is obtained a (voltage × current) or power equation. Thus

$$\frac{q}{C}\frac{dq}{dt} = E\frac{dq}{dt} \quad \text{(power equation)} \tag{3}$$

where the lower-case q indicates that we are now recognizing that the charge Q on the parallel plates of Fig. 1 may vary with time.

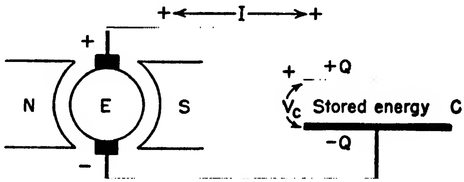


FIG. 1. Illustrating $V_c = Q/C$ as a countervoltage in an electrical circuit.

If the above power equation is considered over any increment of time (say Δt), we have

$$\underbrace{\int_0^{\Delta t} \frac{q}{C}\frac{dq}{dt}\,dt}_{\text{stored energy}} = \underbrace{\int_0^{\Delta t} E\frac{dq}{dt}\,dt}_{\text{supplied energy}} \quad \text{(energy equation)} \tag{4}$$

or

$$\frac{1}{C}\int_0^Q q\,dq = \int_0^Q E\,dq \tag{5}$$

We recognize the right-hand member of equation (5) as the EQ energy which is supplied by generator E, and we may evaluate the left-hand side as

$$\frac{1}{C}\int_0^Q q\,dq = \frac{1}{C}\left[\frac{q^2}{2}\right]_0^Q = \frac{Q^2}{2C}$$

and since $Q^2 = C^2V_c^2$, it follows that

$$\text{energy stored in } C = \frac{CV_c^2}{2} \quad \text{joules} \tag{6}$$

where C is expressed in farads and V_c in volts.

2. Capacitance of Parallel Plates. It has been shown in Section 10 of Chapter VII that the energy stored per unit volume between parallel plates that are equally and oppositely charged (in free space) is

$$w = \frac{\epsilon_0 \mathcal{E}^2}{2} \quad \text{joules/cu m} \tag{7}$$

where $\mathcal{E}$ is the electric field strength in volts/per meter

ϵ_0 is the permittivity of free space $[1/(36\pi \times 10^9)]$ in rationalized mks units.

The magnitude of $\mathcal{E}$ in the region between the parallel plates shown in Fig. 1 (which are illustrated in detail in Fig. 9, page 173) is

$$\mathcal{E} = \frac{V_c}{d} \tag{8}$$

where d is the distance of separation of the plates.

If the plates have surface areas of A $(= a \times b)$ as shown in Fig. 9, page 173, the total energy stored in the region between the plates is

$$W = \frac{\epsilon_0 \mathcal{E}^2 Ad}{2} = \frac{\epsilon_0 V_c^2}{2d^2} Ad \tag{9}$$

Since W has been shown in equation (6) to be $CV_c^2/2$, we have

$$\frac{CV_c^2}{2} = \frac{\epsilon_0 V_c^2 A}{2d} \quad \text{or} \quad C = \frac{\epsilon_0 A}{d} \quad \text{(in free space)} \tag{10}$$

Thus it is shown that the capacitance of parallel plates in free space (or air) is directly proportional to the cross-sectional area A of the intervening dielectric and inversely proportional to the thickness (or $\mathcal{E}$ direction dimension) of the dielectric.

If a material dielectric is employed between the parallel plates, the bound molecular charges (as shown in Fig. 14, page 180) increase the charge Q that can accumulate on the conducting plates for a specified value of $V_c = \mathcal{E}\, d$. Since $\mathcal{E} = \mathbf{D}/\epsilon_0\epsilon_r$ and since $D = \psi/A = Q/A$ in the region between the plates

$$V_c = \frac{D}{\epsilon_0\epsilon_r} d = \frac{Q}{\epsilon_0\epsilon_r A} d$$

and

$$C = \frac{Q}{V_c} = \frac{\epsilon_0\epsilon_r A}{d} \quad \text{farads} \quad \text{(for material dielectrics)} \tag{11}$$

where ϵ_r is the relative permittivity of the dielectric material. (See Table I, page 179.)

Any device consisting of two conducting plates (separated by a dielectric) which is designed as a storage space for electrical potential energy is called a *condenser* or *capacitor*. Many condensers (or capacitors) are of the parallel-plate type, the plates usually being made of metal foil and the dielectric either thin sheets of impregnated paper or mica. This type of condenser is usually made of long narrow sheets of the material rolled into a small volume and sealed in some form of metal or plastic container. Various other types of parallel-plate condensers are employed in practice. The tuning condensers in most radio receivers consist of parallel plates having air as the dielectric and, since variable C is desired, one set of plates is movable relative to the other, thereby varying A in equation (11). Parallel-plate capacitors in large metal containers are sometimes used to compensate for certain detrimental effects of inductance in 60-cycle power systems.

The three basic elements or components of all electrical circuits, aside from the generators and motors, are resistors, inductors, and capacitors.

Example. Let it be required to find the capacitance of the parallel-plate arrangement shown in Fig. 2 if the thickness of the dielectric d is 0.02 cm

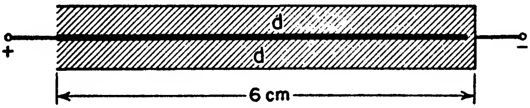


FIG. 2. A parallel-plate capacitor.

and the dimension of the arrangement normal to the page is 3 m. The dielectric is impregnated paper having a dielectric constant (or relative permittivity) of 4.

In Fig. 2 we recognize that the electric flux $\psi = Q$ emanates from both the top and bottom surfaces of the inner conductor. Neglecting the small end effects, the area of the dielectric is

$$A = 2(0.06 \times 3) = 0.36 \quad \text{sq m}$$

$$C = \frac{\epsilon_0 \epsilon_r A}{d} = \frac{1}{36\pi \times 10^9} \times \frac{(4)(0.36)}{0.0002} = 0.0636 \times 10^{-6} \quad \text{farad}$$

Since the farad is an extremely large unit of measure of capacitance, secondary units like microfarads (μf) and the micromicrofarads ($\mu\mu$f) are normally used in practice. For example, the above result might be expressed as

$$0.0636\ \mu\text{f} \quad \text{or} \quad 63{,}600\ \mu\mu\text{f}$$

3. Series and Parallel Arrangements of Capacitors. Fixed capacitors are manufactured in many standard sizes (for example, 0.001 μf, 0.01 μf, 2 μf, and 15 μf), and, to obtain some odd value like 0.128 μf, fixed

capacitors may be arranged either in series or in parallel, and if necessary in series-parallel combinations. Capacitors may also be arranged in series with a view toward increasing the allowable terminal voltage since all capacitors are rated as to the maximum voltage which can be applied to their terminals without injury to the dielectric material.

If the capacitors or condensers are arranged in parallel as shown in Fig. 3, the equivalent two-terminal capacitance at terminals a and b is

$$C_{\text{eq.}} = \frac{Q_{\text{eq.}}}{V_c} = \frac{Q_1 + Q_2 + Q_3}{V_c} = C_1 + C_2 + C_3 \tag{12}$$

Thus the equivalent two-terminal capacitance of any number of capacitors in parallel is the sum of the individual capacitances.

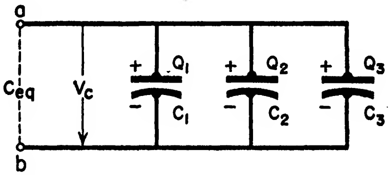


FIG. 3. Capacitors connected in parallel.

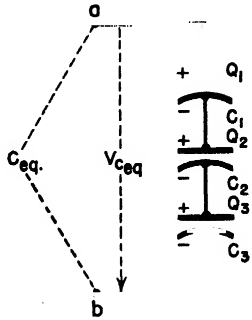


FIG. 4. Capacitors connected in series.

If the capacitors are arranged in series as shown in Fig. 4, the charge passing terminals a and b, $Q_{\text{eq.}}$, is the same charge that appears at each of the condenser plates; that is,

$$Q_{\text{eq.}} = Q_1 = Q_2 = Q_3$$

and since

$$V_{c\text{eq.}} = V_{c1} + V_{c2} + V_{c3}$$

$$C_{\text{eq.}} = \frac{Q_{\text{eq.}}}{V_{c\text{eq.}}} = \frac{1}{\dfrac{V_{c1}}{Q_1} + \dfrac{V_{c2}}{Q_2} + \dfrac{V_{c3}}{Q_3}}$$

or

$$C_{\text{eq.}} = \frac{1}{\dfrac{1}{C_1} + \dfrac{1}{C_2} + \dfrac{1}{C_3}} \quad \text{(for series connection)} \tag{13}$$

Equation (13) may, of course, be extended to include any number of capacitors in series.

Example. Let it be required to find the equivalent two-terminal capacitance of six 0.10-μf capacitors arranged as shown in Figs. 3 and 4 if the a terminals

are connected to form terminal aa and the b terminals are connected to form terminal bb.

$$C_{\text{eq.}\,aa\text{-}bb} = 0.10 + 0.10 + 0.10 + \frac{1}{\dfrac{1}{0.10} + \dfrac{1}{0.10} + \dfrac{1}{0.10}} = 0.333 \quad \mu f$$

In multiple-dielectric cases like that shown in Fig. 5, the guiding principle is that the normal flux-density vector **D** is continuous across

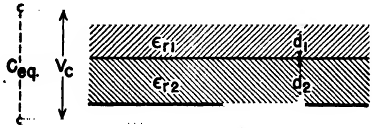


FIG. 5. Cross-section of parallel-plate capacitor having two different dielectrics.

the boundary (or boundaries) between the different dielectric materials. Hence for the parallel-plate arrangement shown in Fig. 5,

$$V_c = \mathcal{E}_1 d_1 + \mathcal{E}_2 d_2 = \frac{D}{\epsilon_0\epsilon_{r1}} d_1 + \frac{D}{\epsilon_0\epsilon_{r2}} d_2 = \frac{Q\,d_1}{\epsilon_0\epsilon_{r1}A} + \frac{Q\,d_2}{\epsilon_0\epsilon_{r2}A} \tag{14}$$

or

$$C_{\text{eq.}} = \frac{Q}{V_c} = \frac{\epsilon_0\epsilon_{r1}\epsilon_{r2}A}{\epsilon_{r2}d_1 + \epsilon_{r1}d_2} = \frac{C_1C_2}{C_1 + C_2} \tag{15}$$

where C_1 and C_2 are the capacitances of the individual layers.

For equal values of d in equation (14), it will be observed that the potential drop ($\mathcal{E}\,d$) is higher across the layer of dielectric having the lower dielectric constant ϵ_r. This principle is made use of in properly proportioning the voltage drops between layers of insulation in the cylindrical bushings of high-voltage transformers.

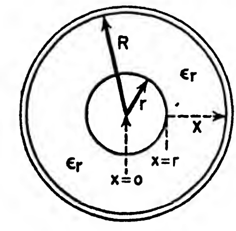


FIG. 6. Coaxial cylindrical conductors.

$$C = \frac{55.55\epsilon_r}{\ln \dfrac{R}{r}}, \quad \mu\mu f/\text{m}.$$

4. Cylindrical-Plate Capacitance. The simplest way of handling electric field problems where long conductors are involved is by the application of Gauss' theorem, illustrated in Fig. 18, page 183. The details of this particular application are given in Section 13 of Chapter VII. As applied to a coaxial cable like that shown in cross-section in Fig. 6, these details are:

(1) Arbitrarily assign $+\sigma$ units of charge per unit length to one conductor and $-\sigma$ to the other.

(2) Determine D_x at any point in the region between the two conductors by Gauss' theorem. The result is

$$D_x = \frac{\sigma}{2\pi x} \quad \text{coulombs/sq m} \tag{16}$$

where D_x is the electric flux density and x is the distance from the center of the inner conductor.

(3) Evaluate the potential difference between the two conductors as

$$V_c = \int_r^R \mathcal{E}_x\, dx = \int_r^R \frac{D_x}{\epsilon_0\epsilon_r}\, dx = \frac{\sigma}{2\pi\epsilon_0\epsilon_r}\int_r^R \frac{dx}{x} = \frac{\sigma}{2\pi\epsilon_0\epsilon_r}\ln\frac{R}{r} \tag{17}$$

Equation (17) is simply a review of Section 13, Chapter VII, and from it we find

$$\frac{V_c}{\sigma} = \frac{18 \times 10^9}{\epsilon_r}\ln\frac{R}{r} \tag{18}$$

The capacitance per unit length of cable follows directly as

$$C_{\text{ul}} = \frac{Q_{\text{ul}}}{V_c} = \frac{\sigma}{V_c} = \frac{\epsilon_r}{(18 \times 10^9)\ln\dfrac{R}{r}} \quad \text{farads/m} \tag{19}$$

or in more convenient form

$$C = \frac{55.55\epsilon_r}{\ln\dfrac{R}{r}} \quad \mu\mu\text{f/m} \tag{19-a}$$

The electric field strength $\mathcal{E}_x$ at any point between the two conductors is

$$\mathcal{E}_x = \frac{D_x}{\epsilon_0\epsilon_r} = \frac{\sigma}{2\pi\epsilon_0\epsilon_r x} \tag{20}$$

and since

$$\sigma = \frac{2\pi\epsilon_0\epsilon_r V_c}{\ln\dfrac{R}{r}} \tag{21}$$

from equation (17), it follows that

$$\mathcal{E}_x = \frac{V_c}{x\ln\dfrac{R}{r}} \quad \text{volts/m} \quad (r < x < R) \tag{22}$$

In a particular case, V_c and $\ln(R/r)$ are fixed quantities and $\mathcal{E}_x$ is observed to vary inversely as the distance from the center of the inner

conductor. It is therefore plain that a dielectric material which can withstand the potential gradient $\mathcal{E}_x$ at the surface of the inner conductor will not be *worked* to capacity at the inner surface of the outer conductor if R is much larger than r, as it is in the lead-out or terminal bushings of some high-voltage apparatus. These bushings are sometimes called condenser bushings because the problem of grading the layers of insulation most economically is essentially a capacitance problem.

Example. *Graded Insulation.* (1) Let it be required to determine the thickness (or $\mathcal{E}$ direction dimension) of the inner layer of dielectric in Fig. 7,

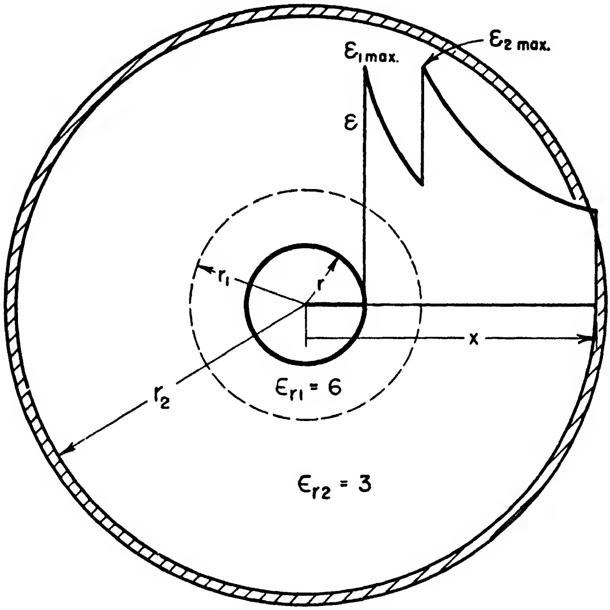


FIG. 7. Graded insulation in a high-voltage condenser bushing.

namely $r_1 - r$, if $\epsilon_{r1} = 6$ and $\epsilon_{r2} = 3$ as indicated, under the specified condition that $\mathcal{E}_{1\,\text{max.}} = \mathcal{E}_{2\,\text{max.}}$, thus subjecting both materials to the same maximum potential gradient.

It will be assumed that, in Fig. 7, $r = 0.5$ cm and $r_2 = 2.4$ cm.

$$\mathcal{E}_{1\,\text{max.}} = \frac{\sigma}{2\pi\epsilon_0\epsilon_{r1}r} = \mathcal{E}_{2\,\text{max.}} = \frac{\sigma}{2\pi\epsilon_0\epsilon_{r2}r_1}$$

The above relations come directly from equation (20) and the principle of continuity of the normal **D** vectors crossing boundaries. It follows that, for $\mathcal{E}_{1\,\text{max.}} = \mathcal{E}_{2\,\text{max.}}$,

$$\epsilon_{r2}r_1 = \epsilon_{r1}r \quad \text{or} \quad r_1 = \tfrac{6}{3} \times 0.5 = 1.0 \quad \text{cm}$$

(2) Assume that the potential difference between the inner and outer conductors is 60,000 volts, and let it be required to find the voltage drop across each of the layers of dielectric material.

This problem may be worked in several ways, but one of the simplest methods is to consider the two layers of dielectric to be two condensers in series, say C_1 and C_2 having dielectric constants of 6 and 3 respectively.

From equation (19),

$$C_1 = \frac{\epsilon_{r1}}{(18 \times 10^9) \ln \dfrac{r_1}{r}} = \frac{6}{(18 \times 10^9) \ln 2} = \frac{1}{(3 \times 10^9)(0.693)}$$

$$C_1 = 4.81 \times 10^{-10} \quad \text{farad/m}$$

$$C_2 = \frac{\epsilon_{r2}}{(18 \times 10^9) \ln \dfrac{r_2}{r_1}} = \frac{3}{(18 \times 10^9) \ln 2.4} = \frac{1}{(6 \times 10^9)(0.8755)}$$

$$C_2 = 1.905 \times 10^{-10} \quad \text{farad/m}$$

Since C_1 and C_2 are in series across the 60,000 volts,

$$C_{eq} = \frac{C_1C_2}{C_1 + C_2} = \frac{(4.81)(1.905)(10^{-20})}{(4.81 + 1.905)(10^{-10})} = 1.364 \times 10^{-10} \quad \text{farad/m}$$

The charge transfer through all seriesed condensers is the same, and in this case

$$\sigma = V_{c\,eq} C_{eq} = V_{c1}C_1 = V_{c2}C_2$$

Hence

$$V_{c1} = \frac{C_{eq}}{C_1} V_{c\,eq} = \frac{1.364}{4.81} \times 60{,}000 = 17{,}000 \quad \text{volts}$$

$$V_{c2} = \frac{C_{eq}}{C_2} V_{c\,eq} = \frac{1.364}{1.905} \times 60{,}000 = 43{,}000 \quad \text{volts}$$

The 17,000 volts and the 43,000 volts are, respectively, the areas under the $\int_r^{r_1} \mathcal{E}_1\, dx$ and $\int_{r_1}^{r_2} \mathcal{E}_2\, dx$ curves shown in Fig. 7.

5. Parallel-Wire Capacitance. In equation (28), page 186, it has been shown that the voltage between two parallel wires which are charged respectively with $+\sigma$ and $-\sigma$ coulombs/m of axial length is

$$V_{ab} = \frac{(36 \times 10^9)\sigma}{\epsilon_r} \ln \frac{D - r}{r} \quad \text{volts} \tag{23}$$

where D is the center-to-center separation of the line wires and r is the radius. The capacitance of the two line wires (shown in Fig. 20, page

185) is therefore

$$C_{\text{w-to-w}} = \frac{\sigma}{V_{ab}} = \frac{\epsilon_r}{(36 \times 10^9) \ln \dfrac{D - r}{r}} \quad \text{farads/m} \tag{24}$$

or, in more convenient form,

$$C_{\text{w-to-w}} = \frac{27.78\epsilon_r}{\ln \dfrac{D - r}{r}} \quad \mu\mu\text{f/m} \tag{24-a}$$

where D is the center-to-center separation of the parallel wires
r is the radius of the cylindrical wires
ϵ_r is the relative permittivity of the electric flux paths.

Another form of equation (24) which is convenient in connection with long lines is

$$C_{\text{w-to-w}} = \frac{0.0194\epsilon_r}{\log \dfrac{D - r}{r}} \quad \mu\text{f/mile} \tag{24-b}$$

which derives directly from equation (24) by recognizing that

$$\frac{\text{(No. of) m}}{\text{mile}} = 1610 \quad \text{and} \quad \ln \frac{D - r}{r} = 2.303 \log \frac{D - r}{r}$$

The capacitance of transmission line wires is an important characteristic of these lines because it is a direct measure of the line as a storage space for electrical potential energy, $CV_{ab}^2/2$. As a result of capacitance, a long high-voltage transmission line may draw from the a-c generator a current which is in the order of 100 amp even though there is no load of any kind connected to the line.

Example. Let it be required to find the wire-to-wire capacitance of 100 miles of No. 0000 B. & S. gage copper wires spaced 8 ft center-to-center in air.

From the wire table,

$$r = \frac{0.460}{2} = 0.23 \quad \text{in.}$$

From equation (24-b),

$$C_{\text{w-to-w}} = 100 \frac{0.0194}{\log \left(\dfrac{96 - 0.23}{0.23} \right)} = \frac{1.94}{\log 416} = \frac{1.94}{2.62} = 0.74 \quad \mu\text{f}$$

6. Single Wire-to-Ground Capacitance by the Method of Images. Conductors are often charged relative to "ground" potential. The ground potential plane may be moist earth itself in the case of power

line wires, or it may be a relatively large metal object which contains a vast amount of free charges in the form of electrons. An example of the latter case is the metal chassis of a radio receiver which acts as an effective *ground* (or uniform potential surface) relative to the minute charge separations which occur at the small radio-frequency voltages which are present in the "live" or non-grounded conductors.

A single wire like that shown in cross-section in Fig. 8 may be operated at a potential difference relative to ground, and in certain cases the capacitance of this wire *relative to ground* may be a very important factor in determining the behavior of the circuit of which the wire is a part. If, for example, the wire is one of the "live" conductors of a radio

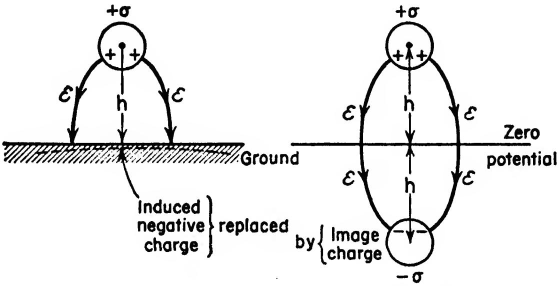


FIG. 8. Illustrating the method of images.

receiver, its capacitance relative to the chassis is called *stray* capacitance, and this type of capacitance must be seriously reckoned with in the design of the radio receiver.

In Fig. 8, the ground plane is viewed as a vast zero-potential conducting plane which is capable of supplying the *induced* charge which the "live" wire attracts. The $+\sigma$ line charge and the induced negative charge of Fig. 8 represent the actual charge distributions of the system. The effect of this distribution of charges in the region between the "live" conductor and ground is the same as though the induced charge were replaced with an *image* charge of $-\sigma$ units of charge per unit length at a distance h below the ground surface. Thus the positive line charge $+\sigma$, h distance above the ground, acting in conjunction with the hypothetical negative line charge $-\sigma$, h distance below the ground, will cause the plane midway between the line charges to be a neutral or ground plane. Since this satisfies the condition imposed by the actual presence of the ground plane, the substitution of the *image* charge for the induced ground charge is a valid substitution provided we confine our investigation after the substitution is made to the region between the real line charge and the ground plane.

Replacing the induced charge indicated in Fig. 8 by the image charge reduces the problem of finding the wire-to-ground capacitance to the parallel-wire case considered in the preceding section. In finding the wire-to-ground capacitance we recognize that the voltage between the conductor to ground is one-half the voltage wire-to-wire of the parallel wire case. Here

$$V_c = V_{\text{w-to-g}} = \frac{V_{ab}[\text{of equation (23)}]}{2}$$

and

$$C_{\text{w-to-g}} = \frac{\sigma}{V_{\text{w-to-g}}} = \frac{\epsilon_r}{(18 \times 10^9) \ln \dfrac{2h - r}{r}} \quad \text{farads/m} \tag{25}$$

or

$$C_{\text{w-to-g}} = \frac{55.55\epsilon_r}{\ln \dfrac{2h - r}{r}} \quad \mu\mu\text{f/m} \tag{25-a}$$

$$C_{\text{w-to-g}} = \frac{0.0388\epsilon_r}{\log \dfrac{2h - r}{r}} \quad \mu\text{f/mile} \tag{25-b}$$

where h is the distance between the line wire and ground
r is the radius of the line wire.

Example. Let it be required to find the capacitance of 1000 ft of No. 12 AWG copper wire relative to the earth if the height of the wire above the ground plane (moist earth in this case) is 20 ft. $\epsilon_r = 1$.

Equation (25-b) is the most convenient form of the derived results to employ in this case. Since 1000 ft $= 1/5.28$ mile,

$$C_{\text{w-to-g}} = \frac{1}{5.28} \times \frac{0.0388}{\log \dfrac{480.0}{0.0404}} = \frac{0.00734}{\log 11{,}890} = \frac{0.00734}{4.075} = 0.0018 \quad \mu\text{f}$$

7. Capacitance of Parallel Wires in the Presence of Ground. The presence of a ground potential surface increases the wire-to-wire capacitance over that given by equation (24). Since both conductor a and conductor b in Fig. 9 have capacitance relative to ground, the effect of the ground is to increase the C_1 capacitance wire-to-wire by an amount $C_2C_3/(C_2 + C_3)$. It should be noted, however, that C_2 and C_3 cannot be evaluated separately by the method given in the preceding section for a single charged wire in the presence of ground.

In order to evaluate the wire-to-wire capacitance in Fig. 9, we replace the induced ground charges due to $+\sigma_a$ and to $-\sigma_b$ in Fig. 9 by image

line charges ($-\sigma_a$ and $+\sigma_b$ respectively) as shown in Fig. 10. The real $+\sigma_a$ and $-\sigma_b$ line charges are, of course, the charges per unit length of line which have been arbitrarily assigned to the conductors for the purpose of finding the potential difference which exists between con-

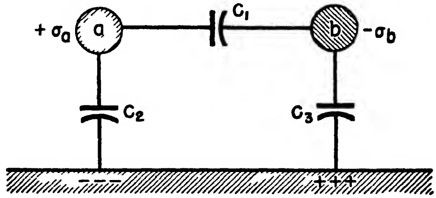


FIG. 9.

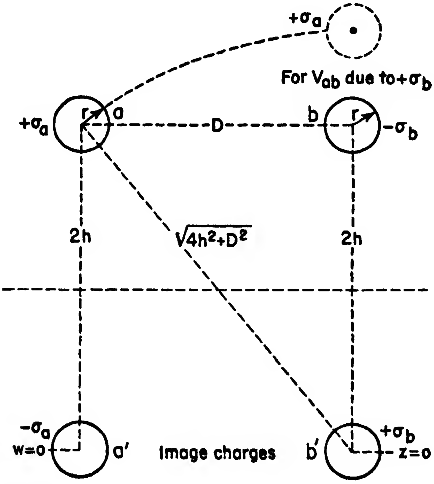


FIG. 10. Evaluation of $C_{\text{w-to-w}}$ in the presence of a plane by the method of images.

ductor a and conductor b. This potential difference V_{ab} is determined with due regard for the electric fields which are established by all four line charges shown in Fig. 10.

Assuming that $D \gg r$, that $2h \gg r$, and that $\epsilon_r = 1$, the voltage drop *from a to b* is the sum of the following four component voltage drops:

$$(1) \qquad V_{ab(\text{due to real } +\sigma_a)} = \frac{\sigma}{2\pi\epsilon_0}\int_r^D \frac{dx}{x} = (18 \times 10^9)\sigma \ln \frac{D}{r} \quad \text{volts} \qquad (26)$$

$$(2) \qquad V_{ab(\text{due to real } -\sigma_b)} = \frac{-\sigma}{2\pi\epsilon_0}\int_D^r \frac{dy}{y} = (-18 \times 10^9)\sigma \ln \frac{r}{D} \quad \text{volts} \qquad (27)$$

where the origin of x in equation (26) is plainly at the center of conductor a, and the origin of y in equation (27) is at the center of conductor b. In finding the component voltage drop *from* conductor a *to* conductor b due to the $+\sigma_b$ image charge, we shift the origin to the center of the $+\sigma_b$ image charge and integrate $\mathcal{E}_z\, dz$ *from* $z = \sqrt{4h^2 + D^2}$ *to* $z = 2h$ along a straight-line path; since, owing to cylindrical symmetry of the electric field established by $+\sigma_b$, conductor a may be moved along the $\sqrt{4h^2 + D^2}$ circular arc shown in Fig. 10 for the purposes of this calculation. Thus

$$(3)\quad V_{ab(\text{due to image } +\sigma_b)} = (18 \times 10^9)\sigma \int_{\sqrt{4h^2+D^2}}^{2h} \frac{dz}{z}$$

$$= (18 \times 10^9)\sigma \ln \frac{2h}{\sqrt{4h^2 + D^2}} \tag{28}$$

and in a similar manner

$$(4)\quad V_{ab(\text{due to image } -\sigma_a)} = (-18 \times 10^9)\sigma \int_{2h}^{\sqrt{4h^2+D^2}} \frac{dw}{w}$$

$$= (-18 \times 10^9)\sigma \ln \frac{\sqrt{4h^2 + D^2}}{2h} \tag{29}$$

The actual voltage drop between conductor a and conductor b for the assumed $+\sigma$ and $-\sigma$ line charges is the sum of (26), (27), (28), and (29) or

$$V_{ab} = (18 \times 10^9)\sigma \left(2 \ln \frac{D}{r} + 2 \ln \frac{2h}{\sqrt{4h^2 + D^2}}\right)$$

$$= (36 \times 10^9)\sigma \ln \frac{D2h}{r\sqrt{4h^2 + D^2}} \tag{30}$$

and

$$C_{\text{w-to-w}} = \frac{\sigma}{V_{ab}} = \frac{1}{(36 \times 10^9) \ln \dfrac{D}{r\sqrt{1 + (D^2/4h^2)}}} \quad \text{farads/m} \tag{31}$$

or

$$C_{\text{w-to-w}} = \frac{0.0194}{\log \dfrac{D}{r\sqrt{1 + (D^2/4h^2)}}} \quad \mu\text{f/mile} \tag{31-a}$$

It will be observed that if $4h^2$ in equation (31) is much larger than D^2, equation (31) reduces to the form shown in equation (24) for $C_{\text{w-to-w}}$ under the conditions that $\epsilon_r = 1$ and that $D \gg r$.

Example. Let it be required to find the per unit increase in the wire-to-wire capacitance of the No. 0000 conductors in the example in Section 5 if these conductors are 10 ft above the ground plane.

The per unit increase is, from equations (24-b) and (31-a),

$$\frac{\log \dfrac{D}{r}}{\log \dfrac{D}{r\sqrt{1 + (D/2h)^2}}} = \frac{\log \dfrac{96}{0.230}}{\log \dfrac{96}{0.230\sqrt{1 + (8/20)^2}}} = \frac{2.620}{2.588} = 1.013$$

The presence of the ground in this case does not increase the wire-to-wire capacitance by much more than 1 per cent; but where the two conductors

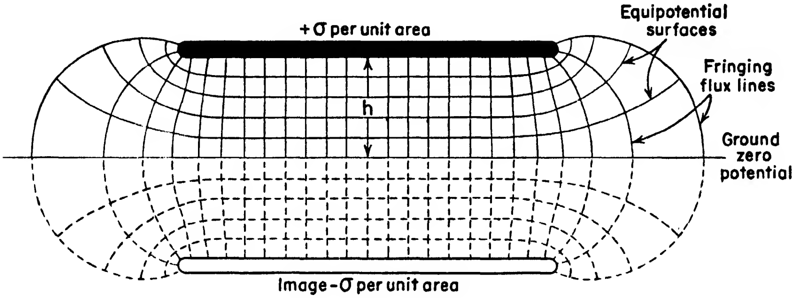


FIG. 11. Electric field map in region near flat charged plate. Solid lines represent map to ground; dashed lines, a continuation to the image conductor.

are in close proximity to a metal chassis, as they might be in a radio receiver or in other high-frequency apparatus, the presence of the ground plane would become much more significant.

8. Evaluation of Capacitance from an Electric Field Map. An electric field map between a flat conductor and a ground plane and between this conductor and its image is shown in Fig. 11. It is assumed that the conductors (the upper conductor and its image) are long in the direction which is normal to the drafting plane; the two-dimensional map shown in Fig. 11 being not too close to either end. Let it be required to calculate the capacitance of a 2-m length of the conductor relative to ground.

Consider an elementary parallelepiped which is bounded on one end by *any one of the curvilinear squares* shown in Fig. 11 and has an axial length (or dimension into the page) of 1 m. From equation (11), the capacitance of *any one* of these elementary parallelepipeds, viewed as a capacitor having parallel plates coinciding with the equipotential boundary surfaces, is

$$C_{\text{el.}} = \frac{\epsilon_0 \epsilon_r A_{\text{el.}}}{d_{\text{el.}}} = \frac{\epsilon_0 \epsilon_r (a_{\text{el.}} \times 1)}{l_{\text{el.}}} = \epsilon_0 \epsilon_r \quad \text{farads} \tag{32}$$

where $a_{\text{el.}}$ is the equipotential dimension of any curvilinear square
$l_{\text{el.}}$ is the $\mathcal{E}$ dimension of the curvilinear square
$a_{\text{el.}}/l_{\text{el.}} = 1$, since it is on this basis that the field map is drawn.

In light of equation (32), any correctly drawn field map may be viewed as the cross-section of a region which is filled with elemental capacitors, a countable number of which are in series, and another countable number of which are in parallel between the two conductors whose capacitance is being evaluated.

In Fig. 11, for example, we count 10 elemental capacitors in series between the upper conductor and its image, or 5 in series between the upper conductor and ground. In the mapped portion (of the total volume of the electric field which exists between the upper conductor and ground) we count 24 tubes of electric flux which are in parallel. In this connection, a tube of flux is bounded by the flow lines (or vertical lines in Fig. 11). There are, therefore, 24 parallel stacks of series (elemental) capacitors in each meter length of the conductor, or a total of 48 parallel stacks in the 2-m length of conductor specified. (Since mks units are to be employed in the final evaluation, the meter is taken to be a unit length.)

Considering only the mapped portion of the field shown in Fig. 11,

$$C_{\text{plate-to-grd.}} = \epsilon_0\epsilon_r \frac{\text{(No. of) parallel stacks}}{\text{series elements/stack}}$$

$$= \frac{\epsilon_r}{36\pi \times 10^9} \times \frac{48}{5} = 0.0848\,\epsilon_r \times 10^{-9} \quad \text{farad}$$

or

$$C_{\text{plate-to-grd.}} = 84.8\epsilon_r \quad \mu\mu\text{f} \tag{33}$$

The above value of capacitance is appreciably larger and more accurate than if the fringing flux shown in Fig. 11 had been neglected and the parallel-plate formula [equation (11)] used directly. Neglecting fringing, the result would have been

$$C = \frac{\epsilon_r}{36\pi \times 10^9} \times \frac{2(20a_{\text{el.}})}{5l_{\text{el.}}} \quad \text{farad} \quad (\text{or } 70.7\epsilon_r\ \mu\mu\text{f}) \tag{34}$$

which is significantly smaller than that obtained in equation (33).

One fact common to both equations (32) and (34) is that the capacitance has been evaluated in each case without knowing specifically the actual values of $a_{\text{el.}}$ and $l_{\text{el.}}$, since only the ratio of these dimensions is necessary in the evaluation of capacitance; and this ratio is fixed as unity by the mapping process.

There is actually more fringing between the upper surface of the

upper conductor and ground in Fig. 11 than can be shown on an illustration as small as Fig. 11. By extending the upper four equipotential lines shown in Fig. 11 upward and around to encircle the $+\sigma$ conductor, it is not difficult to show that approximately 6 or more tubes of flux (or 6 more parallel paths) are actually present between the upper conductor and ground. And the total number of parallel paths determines the actual capacitance between conductors or between a conductor and ground. Hence a more accurate value of the capacitance between the upper conductor (shown in Fig. 11) and ground than that given in equation (33) is

$$C_{\text{plate-to-grd.}} = \frac{(24 + 6)}{24} 84.8\epsilon_r = 106\epsilon_r \quad \mu\mu\text{f} \tag{35}$$

Even though the factor 6 in the above equation is in error by 15 or 20 per cent, this error will affect the result by only about 3 per cent, since the 24 parallel flux paths which have been mapped more accurately (than the estimated 6) represent the controlling factor in equation (35).

The field mapping method of determining capacitance, in the hands of a good field mapper, will give results which, although still approximate, are the most accurate obtainable where conductors of irregular shapes are involved. Emphasis here is placed on the elementary theory which underlies the field mapping method of determining capacitance rather than upon the technique of drawing the maps. This technique can be acquired if the theory is understood.

9. Capacitance Analogous to Resilience Constant of a Spring. In Section 9 of Chapter XII, an analogy was drawn between self-inductance in the electromagnetic system and mass in the mechanical system. By extending this analogy, it is found that capacitance is analogous to the resilience constant of a spring as illustrated in Fig. 12. The resilience constant of a spring, K_r, is defined as the elongation of the spring per unit force applied to the spring. Thus

$$K_r = \frac{l}{F_r} \quad \text{as} \quad C = \frac{Q}{V_c}$$

In the mechanical system shown in Fig. 12-a, the three counter or restraining forces are f_m (the inertial force), f_d (the frictional force), and f_r (the resilience force developed by the spring). Hence

$$f_m + f_d + f_r = f \quad \text{(applied)} \tag{36}$$

or

$$m\frac{dv}{dt} + K_d v + \frac{l}{K_r} = f \quad \text{(applied)} \tag{36-a}$$

In the electrical system shown in Fig. 12-b, the three countervoltages are v_L (the voltage of self-inductance), v_R (the resistive voltage drop), and v_C (the counter voltage developed by the capacitor).

$$v_L + v_R + v_C = e \quad \text{(applied)} \tag{37}$$

or

$$L\frac{di}{dt} + Ri + \frac{q}{C} = e \quad \text{(applied)} \tag{37-a}$$

Since the governing differential equations of the two systems, (36-a) and (37-a), are of precisely the same form, there exists a direct or one-

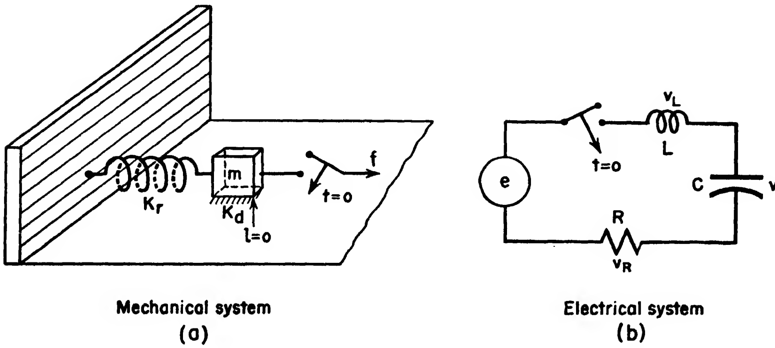


Fig. 12. The analogy between capacitance C and resilience constant K_r.

to-one correspondence between the two systems which permits an interchange of solutions and even a quantitative substitution of one system for the other in the laboratory.

The electrical potential energy stored in the capacitor is directly analogous to the mechanical potential energy stored in the spring. The potential energy stored in the charged capacitor is

$$W_c = \int_0^Q v_C\,dq = \int_0^Q \frac{q}{C}\,dq = \frac{Q^2}{2C} = \frac{CV_c{}^2}{2} \tag{38}$$

and the potential energy stored in the stretched spring is

$$W_r = \int_0^{L'} f_r\,dl = \int_0^{L'} \frac{l}{K_r}\,dl = \frac{L'^2}{2K_r} = \frac{K_rF_r{}^2}{2} \tag{39}$$

where L' is the particular elongation of the spring produced by some particular value of steady force F_r applied to the spring.

A tabulation of the various electrical and mechanical quantities which appear in the governing differential equations of the two systems shown in Fig. 12 is given in Table I. The dimensional correspondence between

the various derived quantities in terms of the fundamental quantities (mass, length, and time) is also indicated in Table I.

TABLE I

Electrical			Mechanical		
Quantity	Symbol	Dimensions	Quantity	Symbol	Dimensions
Inductance	L		Mass	m	
Charge	Q		Length	l	
Time	t		Time	t	
Voltage	V	$= L^1Q^1t^{-2}$	Force	f	$= m^1l^1t^{-2}$
Current	i	$= Q^1t^{-1}$	Velocity	v	$= l^1t^{-1}$
Resistance	R	$= L^1t^{-1}$	Damping constant	K_d	$= m^1t^{-1}$
Capacitance	C	$= L^{-1}t^2$	Resilience constant	K_r	$= m^{-1}t^2$
Energy	$W_{\text{el.}}$	$= L^1Q^2t^{-2}$	Energy	$W_{\text{me.}}$	$= m^1l^2t^{-2}$

For the present, no attempt will be made to solve the general differential equations that govern the dynamic behavior of the two systems shown in Fig. 12. Our first goal is that of investigating the simple RC circuit which is a special case of that shown in Fig. 12-b; namely, the case where $v_L = L\,(di/dt)$ is negligibly small as compared with $v_C = q/C$. Equation (37-a) then reduces to

$$Ri + \frac{q}{C} = e \quad \text{(applied)}\ . \tag{40}$$

The corresponding situation in Fig. 12-a would be that where $f_m = m\,(dv/dt)$ was negligibly small as compared with $f_r = l/K_r$, as would be the case if the mass were sufficiently small and the spring were sufficiently weak (high value of K_r). A weak spring (high K_r) will store more potential energy per unit of applied force than will a stronger spring (low K_r); and analogously a high value of C will store more potential energy per unit of applied voltage than will a lower value of C.

10. Return of Stored Potential Energy to the Electrical Circuit. In Fig. 13, it is assumed that the capacitor C has been charged to a voltage $V_c = Q_0/C$ prior to $t = 0$. The charge Q_0 might have been present at the plates of the capacitor for some period of time prior to the closing of switch S in Fig. 13, since a perfect dielectric material is assumed.

At $t = 0$, the switch in Fig. 13 is closed and the positive charge and negative charge, in recombining through the closed circuit, produce a current through resistor R which is directed oppositely to the $+i$ shown on the circuit diagram. (In this case the $+i$ refers to the direction of the *charging* current which, when flowing under the influence of the original

charging voltage V_c, charges the condenser with the polarity indicated on the diagram.)

At and after $t = 0$, equation (40) takes the form

$$Ri + \frac{q}{C} = 0 \tag{41}$$

subject to the condition that $q = Q_0$ at $t = 0$. It will be recognized that the symbol q in equation (41) refers to the charge which is present at

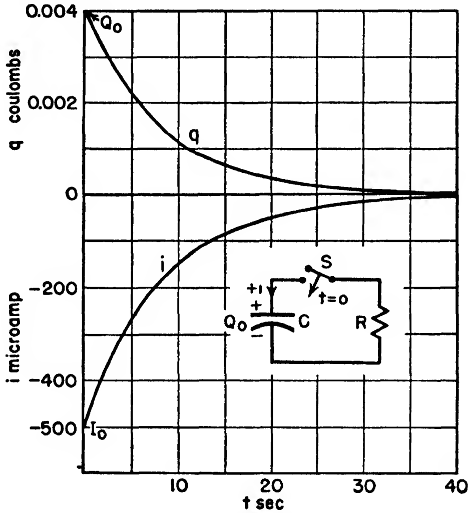


FIG. 13. Charge and current variations for $R = 2 \times 10^6$ ohms, $C = 4 \times 10^{-6}$ farad.

the condenser plates; but any charge that leaves the condenser plates must flow through R and hence $dq/dt = i$ in equation (41). Hence equation (41) may be written as

$$R\frac{dq}{dt} + \frac{q}{C} = 0 \tag{42}$$

A solution of the above equation for q as a function of time is

$$q = A\varepsilon^{\alpha t} \tag{43}$$

since

$$R(A\alpha\varepsilon^{\alpha t}) + \frac{A}{C}\varepsilon^{\alpha t} = A\varepsilon^{\alpha t}\left(R\alpha + \frac{1}{C}\right) = 0 \quad \text{(at all } t\text{)}$$

provided only that

$$R\alpha + \frac{1}{C} = 0 \quad \text{or} \quad \alpha = -\frac{1}{RC} \tag{44}$$

The constant A may be evaluated by imposing the initial condition ($q = Q_0$ at $t = 0$) on equation (43), which tells us that $A = Q_0$. Thus

$$q = Q_0 \varepsilon^{-t/RC} \tag{45}$$

and the circuit current

$$i = \frac{dq}{dt} = -\frac{Q_0}{RC}\varepsilon^{-t/RC} \tag{46}$$

Equation (45) shows that the charge at the plates of the capacitor subsides from its initial value of Q_0 at a time rate which is governed by the RC product. Equation (46) shows that the current at $t = 0$ is

$$I_0 = -\frac{Q_0}{RC} = -\frac{V_{c0}}{R}$$

where V_{c0} is the initial capacitor voltage and the minus sign refers to the fact that the current through resistor R on discharge is opposite to the direction of positive charging current.

Time Constant. The length of time required for q (or i) to make 63.2 per cent of its total change is RC, since it is this value of time which, when entered into equation (45), reduces the charge q to $0.368Q_0$. In the present case, the charge q changes from Q_0, at $t = 0$, to zero value at $t = \infty$. Thus when q has been reduced to $0.368Q_0$, 63.2 per cent of the total change has been made.

The product RC is, therefore, the time constant of the RC circuit. That this product is dimensionally equivalent to time may be seen from Table I.

Example. The current-time graph shown in Fig. 13 is based on the following data:

$$V_{c0} = 1000 \text{ volts} \quad C = 4.0 \times 10^{-6} \text{ farad} \quad R = 2 \times 10^6 \text{ ohms}$$

$$Q_0 = CV_{c0} = (4 \times 10^{-6})(10^3) = 0.004 \text{ coulomb}$$

$$I_0 = -\frac{Q_0}{RC} = -\frac{V_{c0}}{R} = -\frac{1000}{2 \times 10^6} = -500 \times 10^{-6} \text{ amp} \quad (\text{or } -500\ \mu\text{amp})$$

At $t = 10$ sec,

$$i = -I_0\,\varepsilon^{-t/RC} = -500\,\varepsilon^{-10/8} = -\frac{500}{3.49} = -143 \quad \mu\text{amp}$$

Other points on the current-time graph are found in a similar manner.

The time constant for this particular circuit is $RC = 8$ sec.

Time variations of condenser charge as shown in Fig. 13 are used in many electronic devices to obtain time-varying voltages (q/C); and, if only a small portion of the total q variation is employed, a voltage

variation which is reasonably linear with respect to time can be obtained. It will be observed from the q graph shown in Fig. 13 that from $t = 0$ to $t = 2$ sec, for example, the q (and hence q/C) time variation approaches a straight-line variation quite closely.

11. Storing Additional Potential Energy. In Fig. 14, it will be assumed that the capacitor C possesses an initial charge $Q_0 = V_{c0}C$ prior to $t = 0$; and, as a result of this initial charge, there is stored in the electric field between the plates of the capacitor a potential energy

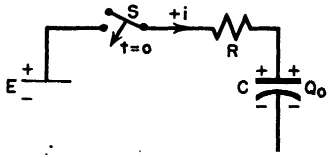


FIG. 14. The RC circuit.

$$\frac{Q_0^2}{2C} = \frac{CV_{c0}^2}{2} \quad \text{(initial stored energy)}$$

When switch S in Fig. 14 is closed, the capacitor accepts more energy from the driving source E if $E > Q_0/C$; and returns energy to the driving source if $Q_0/C > E$.

The basic voltage equation which is applicable to the circuit shown in Fig. 14 at and after $t = 0$ is

$$Ri + \frac{q}{C} = E \quad \text{(where } E \text{ is constant)} \tag{47}$$

subject to the initial condition that $q = Q_0$ at $t = 0$. Since the current i which flows through the resistor R is the time rate of change of the capacitor charge, equation (47) may be written as

$$R\frac{dq}{dt} + \frac{q}{C} = E \tag{48}$$

The solution of this equation takes the same form as that given in equation (39), page 336; that is,

$$q = \underset{\text{steady}}{CE} + \underset{\text{transient}}{A_1\varepsilon^{\alpha t}} \tag{49}$$

where we know physically that the capacitor will ultimately reach a steady-state charge which is equal to CE coulombs.

After substituting q, as given in equation (49), into equation (48), there is obtained

$$RA_1\alpha\varepsilon^{\alpha t} + E + \frac{A_1}{C}\varepsilon^{\alpha t} = E$$

or

$$A_1\varepsilon^{\alpha t}\left(R\alpha + \frac{1}{C}\right) = 0 \quad \left(\text{for all } t \text{ if } \alpha = -\frac{1}{RC}\right)$$

If the initial condition ($q = Q_0$ at $t = 0$) is written into equation (49), we obtain the value of $A_1 = (Q_0 - CE)$. Therefore

$$q = CE + (Q_0 - CE)\,\varepsilon^{-t/RC} \tag{50}$$

and

$$i = \frac{dq}{dt} = \frac{\left(E - \dfrac{Q_0}{C}\right)}{R}\,\varepsilon^{-t/RC} \tag{51}$$

A graph of the q versus t variation as defined by equation (50) will show that the charge at the capacitor plates changes exponentially from Q_0, at $t = 0$, to a new value CE at $t = \infty$. During this transition period the stored energy changes from $Q_0^2/2C$ to $CE^2/2$. During this same period, the current changes from zero value at $t = 0-$ (just before the switch is closed) to $I_{0+} = (E - Q_0/C)/R$ (just after the switch is closed) and then subsides in magnitude as time progresses, finally reaching zero value again when the condenser voltage equals the applied voltage E.

12. Current Continuity in the *RC* Circuit. Thus far we have considered only the current which flows through the external circuit which joins the plates of the capacitor. When current flows in the *RC* loop, it establishes a magnetic field as would any other current-carrying loop, and it becomes desirable to think of the loop current as continuous around the closed loop. Plainly, the current through the dielectric material which forms a portion of the *RC* loop cannot be the same type of conduction current which exists in the metallic portion of the loop, since no free charge carriers are present in a perfect dielectric.

A conduction current i flows in the metallic portion of the *RC* loop only when the capacitor charge q is in the state of change, that is, when dq/dt has a value other than zero. The time rate of change of the charge on the capacitor plates results in a time rate of change of electric flux $d\psi/dt$ between the plates, and since $\psi = q$

$$i_d = \frac{d\psi}{dt} = \frac{dq}{dt} \quad \text{(displacement current)} \tag{52}$$

where q is the accumulated charge at one of the plates of the capacitor. Where a perfect dielectric is employed the displacement current, i_d which flows *through* the dielectric is numerically equal in amperes to the conduction current i which flows in the metallic portion of the *RC* loop. Thus the current is rendered continuous around the closed loop if both conduction and displacement currents are considered.

The concept of displacement current also provides us with a physical

picture of the current relations in a leaky capacitor where an actual conduction current flows through the imperfect dielectric. In Fig. 15, this conduction current is indicated as i_c flowing through R_c, the leakage resistance of the dielectric material. The displacement current i_d also flows from plate to plate through the dielectric material and is therefore in parallel with i_c; and the external circuit current is the sum of i_c and i_d. Since $i_d = dq/dt = C(dv_c/dt)$,

$$i = i_c + i_d = \frac{v_c}{R_c} + C\frac{dv_c}{dt} \tag{53}$$

Under steady v_c conditions ($dv_c/dt = 0$), the displacement current is zero, and the external circuit current is equal to the conduction leakage current v_c/R_c. If v_c is a time-varying voltage, both i_c and i_d contribute to the total circuit current.

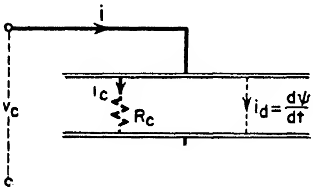


FIG. 15. Illustrating leakage current i_c and displacement current i_d through the dielectric of a capacitor.

The more perfect the dielectric material, the higher will be the ohmic value of R_c and the smaller will be leakage current i_c. All practical capacitors have a finite value of leakage resistance but, where good dielectric materials are used, R_c is so many millions of ohms that i_c becomes negligibly small. In this case the external circuit current is equal to the displacement current.

Example. Consider a leaky capacitor like that shown in Fig. 15 where

$$R_c = 10^6 \text{ ohms} \quad C = 0.2 \times 10^{-6} \text{ farad} \quad v_c = 10 \sin (200t) \text{ volts}$$

Let it be required to find the value of the circuit current i under this condition of operation.

$$i_c = \frac{v_c}{R_c} = \frac{10 \sin (200t)}{10^6} \text{ amp} \quad \text{or} \quad 10 \sin (200t) \quad \mu\text{amp}$$

$$i_d = C\frac{dv_c}{dt} = (0.2 \times 10^{-6})(10 \times 200) \cos (200t) \quad \text{amp}$$

or

$$i_d = 400 \cos (200t) \quad \mu\text{amp}$$

$$i = i_c + i_d = 10 \sin (200t) + 400 \cos (200t) \quad \mu\text{amp}$$

The sine term and the cosine term in the above equation may be combined into a single sine term by the simple trigonometric substitution which makes $\theta = \tan^{-1} 400/10$. Then, plainly,

$$\cos\theta = \frac{10}{\sqrt{10^2 + 400^2}} \quad \text{and} \quad \sin\theta = \frac{400}{\sqrt{10^2 + 400^2}}$$

and

$$i = \sqrt{10^2 + 400^2}\left[\frac{10}{\sqrt{10^2 + 400^2}}\sin(200t) + \frac{400}{\sqrt{10^2 + 400^2}}\cos(200t)\right]$$

$$= 400.1\,[\cos\theta\sin(200t) + \sin\theta\cos(200t)]$$

$$= 400.1\sin(200t + 88.57°) \quad \mu\text{amp}$$

In this case the leakage current i_r increases the magnitude of the circuit current by only about 1 or 2 parts in 4000 over the value which would exist if R_c had been infinitely large.

13. Steady-State Current in the *RC* Circuit with A-C Driving Voltage. If the battery voltage E in Fig. 14 is replaced with a sinusoidally varying voltage, $e = E_m \sin(\omega t)$ volts, the basic voltage equation for the RC loop becomes

$$Ri + \frac{q}{C} = E_m \sin\omega t \tag{54}$$

where E_m is the maximum magnitude of the a-c driving voltage
$\omega = 2\pi f$ is the angular frequency in radians per second.

After all transient disturbances have subsided we may expect that a sinusoidally varying current will flow in the RC series loop. Our experience with the LR circuit under a-c driving voltage conditions further leads us to suspect that the current wave may be displaced by some time angle, say θ, from the sinusoidal voltage wave. Subject to later proof, we shall write the form of the instantaneous current in the RC series loop as

$$i = I_m \sin(\omega t + \theta) \tag{55}$$

where I_m is the maximum magnitude of the sinusoidally varying current. We also expect that I_m will depend upon the maximum magnitude of the applied voltage E_m.

If $i = I_m \sin(\omega t + \theta)$ is the correct expression for the circuit current and if a perfect dielectric material is used in the capacitor,

$$q = \int i\,dt = -\frac{I_m}{\omega}\cos(\omega t + \theta) + A_1 \tag{56}$$

where the constant of integration A_1 is the magnitude of the transient component of charge which has presumably died away. Hence we let $A_1 = 0$ here since we are interested only in the steady-state alternating current.

The test of whether or not equation (55) is the correct expression for the current in the RC circuit is whether or not $Ri + (q/C)$ will precisely

balance the driving voltage, $E_m \sin (\omega t)$ volts, at all times during the steady-state operation of this circuit. To make this test, we write

$$Ri + \frac{q}{C} = RI_m \sin (\omega t + \theta) - \frac{I_m}{\omega C} \cos (\omega t + \theta) \tag{57}$$

The sine term and the cosine term in the above equation may be combined into a single sine term by the method given in the preceding example. The trigonometric substitution which will effect this combination is that of letting

$$\theta = \tan^{-1} \frac{1}{\omega CR} \tag{58}$$

and hence

$$\cos \theta = \frac{R}{\sqrt{R^2 + \left(\frac{1}{\omega C}\right)^2}} = \frac{R}{Z}$$

$$\sin \theta = \frac{\frac{1}{\omega C}}{\sqrt{R^2 + \left(\frac{1}{\omega C}\right)^2}} = \frac{X_c}{Z}$$

For convenience, $\sqrt{R^2 + (1/\omega C)^2}$ is called Z, the impedance of the RC series circuit; and $1/\omega C$ is called X_c, the capacitive reactance of the circuit.

Employing the trigonometric substitution suggested in equation (58), we find that equation (57) becomes

$$Ri + \frac{q}{C} = ZI_m[\cos \theta \sin (\omega t + \theta) - \sin \theta \cos (\omega t + \theta)]$$

$$= ZI_m \sin \omega t = E_m \sin \omega t \tag{59}$$

Thus it is shown that equation (55) is the correct form for the steady-state alternating current that flows in the RC series circuit and that

$$I_m = \frac{E_m}{Z} = \frac{E_m}{\sqrt{R^2 + \left(\frac{1}{\omega C}\right)^2}} \text{ amp} \tag{60}$$

If R is very small as compared to $1/\omega C$,

$$I_m = \omega C E_m = \frac{E_m}{\frac{1}{\omega C}} \text{ amp} \tag{61}$$

which shows clearly that the maximum magnitude of the current in the RC series circuit increases with both the frequency of the driving voltage and with the magnitude of the capacitance.

Capacitive reactance is by definition

$$X_c = \frac{1}{\omega C} \quad \text{ohms} \tag{62}$$

X_c represents ohms because it is the factor by which the voltage (across the capacitor plates) must be divided to obtain the current which flows

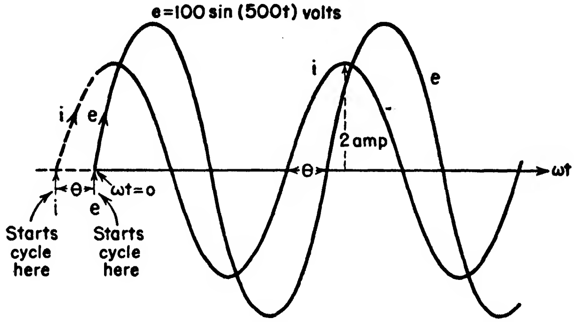


FIG. 16. Illustrating the time lead of the current variation relative to the applied a-c voltage.

through the capacitor. A capacitor of 100 $\mu\mu$f, for example, represents almost an open circuit (infinite impedance) to a 60-cycle driving voltage since

$$X_c = \frac{1}{(2\pi 60)(100 \times 10^{-12})} = 26.5 \times 10^6 \quad \text{ohms}$$

The same capacitor operated at 100 megacycles, the frequency of some radio broadcasting stations, is

$$X_c = \frac{1}{(2\pi 10^8)(100 \times 10^{-12})} = 15.9 \quad \text{ohms}$$

An important characteristic of capacitive reactance is that it causes the circuit current i to *lead* the driving voltage e by the time angle θ as indicated in Fig. 16. Since inductive reactance ωL causes the current to *lag* the driving voltage, $1/\omega C$ and ωL tend to cancel one another when both are present in the series circuit. The cancellation effect of $1/\omega C$ upon ωL in an LRC series circuit like that shown in Fig. 12-b is accounted for by attaching a negative sign to $1/\omega C$ when both types of reactance are present in the series loop. The combined reactance is then $\omega L -$

$(1/\omega C)$. The *LRC* series circuit then behaves either as an *LR* circuit (if $\omega L > 1/\omega C$), or as an *RC* circuit (if $1/\omega C > \omega L$), or as a simple *R* circuit (if $\omega L = 1/\omega C$). In the latter case the reactance cancellation is complete, and the circuit is said to be "in resonance."

Example. Let it be required to construct a diagram of the time variations of the driving voltage e and the circuit current i in an *RC* circuit where

$$R = 30 \text{ ohms} \quad C = 50\ \mu\text{f} \quad e = 100 \sin (500t) \text{ volts}$$

First we note that $\omega = 2\pi f = 500$ radians/sec; making $f = 500/6.28 = 79.6$ cycles/sec.

$$X_c = \frac{1}{\omega C} = \frac{1}{(500)(50 \times 10^{-6})} = 40 \text{ ohms} \quad \text{(capacitive reactance)}$$

$$Z = \sqrt{R^2 + X_c^2} = \sqrt{30^2 + 40^2} = 50 \text{ ohms} \quad \text{(circuit impedance)}$$

$$\theta = \tan^{-1} \frac{X_c}{R} = \tan^{-1} \frac{40}{30} = 53.1° \quad \text{(time angle of lead of } i\text{)}$$

$$i = \frac{E_m}{Z} \sin (\omega t + \theta) = \frac{100}{50} \sin (500t + 53.1°)$$

$$= 2.0 \sin (500t + 53.1°) \quad \text{amp}$$

The maximum magnitude of the current is 2 amp, and the current-time wave *leads* the voltage-time wave, $e = 100 \sin (500t)$ volts, by a time angle of $\theta = 53.1°$ as shown in Fig. 16.

14. Oscillatory Systems. In Fig. 12 are shown two analogous systems, both of which are capable of natural oscillation if shock-excited in any manner. If, for example, a steady force were suddenly applied to the mechanical system shown in Fig. 12-a, this constant force plus the inertial force $m\ (dv/dt)$ would tend to make mass m overshoot its final position or steady-state displacement from $l = 0$. The physical picture may be somewhat clearer if we think of Fig. 12-a as being rotated clockwise through 90° and let the force of gravity be the steady force which is applied when suddenly we remove a support which, prior to $t = 0$, has kept the spring in an unstretched state. Our everyday experience tells us that mass m will then tend to oscillate about its final resting position.

Equation (36-a), which governs the oscillatory motion of mass m in Fig. 12-a, is entirely analogous to equation (37-a). We may, therefore, expect that the *LRC* series circuit is capable of natural oscillation. An oscillographic record of the type of current oscillation which can be obtained in a series *LRC* circuit is shown in Fig. 17. The current oscillation (which is analogous to the velocity oscillation of mass m) is an

exponentially damped sine wave which finally reaches zero when the condenser voltage reaches its steady-state value of $V_c = E$.

After the condenser in Fig. 17 is charged, the LRC series loop may be closed on itself (without battery E being in the loop), and a discharge oscillation identical in magnitude and form to that shown in Fig. 17 will occur as the condenser voltage subsides to zero. Before the advent of the vacuum-tube oscillator, a continuous succession of damped oscilla-

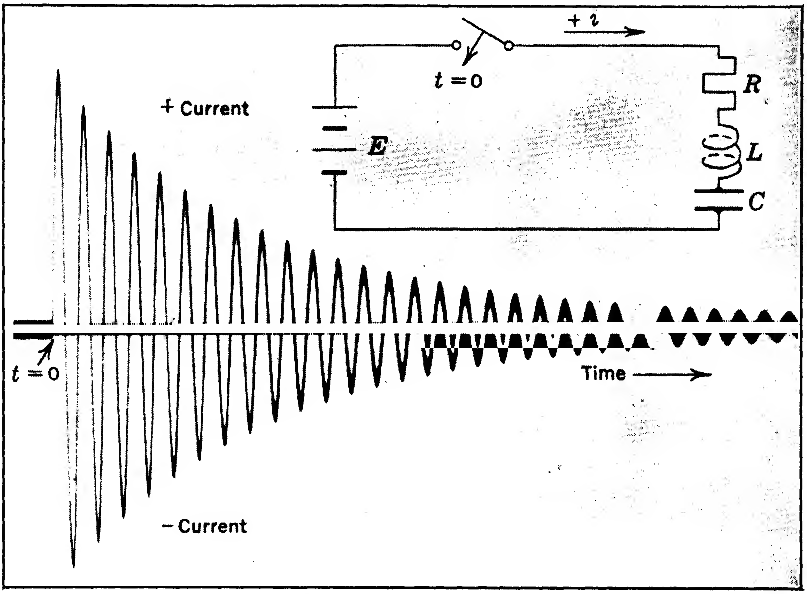


FIG. 17. Illustrating the oscillatory current in an LRC circuit.

tions like those shown in Fig. 17 was employed by the early broadcast stations as the radio-frequency oscillations necessary for the transmission of intelligence through free space. The detailed analysis of the LRC circuit will be encountered in later courses.

15. Summary. (1) Capacitance is defined as

$$C = \frac{Q}{V_c} \text{ farads} \tag{63}$$

(2) Capacitance may be calculated in terms of the dimensions of the conductors and ϵ wherever V_c (which is developed by equal and opposite charges on the conductors) can be evaluated.

$$C = \frac{\epsilon_0 \epsilon_r A}{d} \text{ farads} \quad \text{(for parallel plates and no fringing)} \tag{64}$$

$$C = \frac{55.55\epsilon_r}{\ln \dfrac{R}{r}} \quad \mu\mu\text{f/m} \quad \text{(for concentric cylindrical conductors)} \tag{65}$$

$$C_{\text{w-to-w}} = \frac{27.78\epsilon_r}{\ln \dfrac{D - r}{r}} \quad \mu\mu\text{f/m} \quad \text{(for parallel cylindrical conductors)} \tag{66}$$

$$C_{\text{w-to-grd.}} = \frac{55.55\epsilon_r}{\ln \dfrac{2h - r}{r}} \quad \mu\mu\text{f/m} \quad \text{(for one wire to ground)} \tag{67}$$

(3) Capacitance between conductors of irregular shapes may be determined from electric field maps by considering the elemental capacitors shown on the map as a series-parallel combination of capacitors each having

$$C_{\text{el.}} = \epsilon_0\epsilon_r \quad \text{farads of capacitance} \tag{68}$$

(4) $C_{\text{eq.}} = C_1 + C_2 + C_3 + \cdots$ (for paralleled capacitors) (69)

(5) $C_{\text{eq.}} = \dfrac{1}{\dfrac{1}{C_1} + \dfrac{1}{C_2} + \dfrac{1}{C_3} + \cdots}$ (for seriesed capacitors) (70)

(6) Condenser discharge current $= -\dfrac{Q_0}{RC}\,\varepsilon^{-t/RC}$ amp (71)

(7) Condenser charging current $= \dfrac{\left(E - \dfrac{Q_0}{C}\right)}{R}\,\varepsilon^{-t/RC}$ amp (72)

(8) time constant of the RC circuit $t_c = RC$ (73)

(9) The displacement current *through* the dielectric material of a capacitor is

$$i_d = \frac{d\psi}{dt} = \frac{dq}{dt} \quad \text{amp} \tag{74}$$

(10) The steady-state alternating current in an RC series circuit energized with $e = E_m \sin \omega t$ volts is

$$i = \frac{E_m}{\sqrt{R^2 + \left(\dfrac{1}{\omega C}\right)^2}} \sin\left(\omega t + \tan^{-1}\frac{1}{\omega CR}\right) \quad \text{amp} \tag{75}$$

where $1/\omega C = X_c$ is called the capacitive reactance of the RC circuit.

PROBLEMS

1. Two sheets of metal foil 4 m long and 10 cm wide are separated from one another by impregnated paper which is 0.035 cm thick and which has a relative permittivity of 3.5. What is the capacitance of the resulting capacitor expressed in microfarads?

2. The length of the mica condenser shown in cross-section in Fig. 18 is 4 cm. The mica dielectric is 0.01 cm thick, it has a dielectric constant of 6 and a dielectric strength of 25 kv/mm.

(*a*) What is the capacitance of the condenser expressed in micromicrofarads?

(*b*) What would be the voltage rating of this condenser based on the mica being able to withstand twice rated voltage plus 1000 volts?

3. What is the value of $C_{eq.}$ in Fig. 19 if $C_1 = 1\ \mu f$, $C_2 = 2\mu f$, and $C_3 = 3\mu f$?

4. Given three fixed capacitors: $C_1 = 1\mu f$, $C_2 = 2\mu f$, and $C_3 = 3\mu f$. By using the capacitors either one, two, or three at a time and in any series, parallel, or series-parallel combination, find all the *different* capacitances that can be obtained. Tabulate results in order of increasing magnitudes of equivalent capacitance.

5. Find the capacitance of a pair of parallel conducting plates each of which has an area of 0.2 x 1.5 sq m if they are separated by two layers of dielectric material: (1) paper, 0.002 m thick, the ϵ_r of which is 3; (2) mica, 0.001 m thick, the ϵ_r of which is 6. Express the result in micromicrofarads.

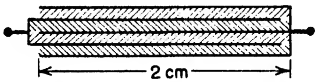


FIG. 18. See Prob. 2.

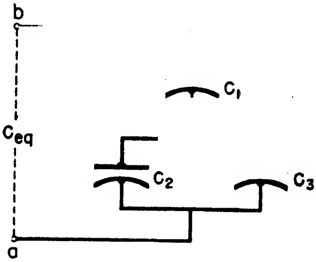


FIG. 19. See Prob. 3.

6. (*a*) Find the capacitance of a pair of parallel plates, each of which has an area of 0.2 x 1.5 sq m if they are separated by the following layers of dielectric material stacked one on top of the other:

Layer 1 – paper, 0.001 m thick, $\epsilon_r = 3.0$
Layer 2 – rubber, 0.002 m thick, $\epsilon_r = 5.0$
Layer 3 – glass, 0.003 m thick, $\epsilon_r = 7.0$

Neglect fringing.

(*b*) If a potential difference of 10,000 volts is maintained across the condenser plates, find the potential difference across each layer of dielectric material.

7. What is the micromicrofarad capacitance per foot of coaxial cable like that shown in Fig. 6, page 358, if $R/r = 4.8$ and a solid dielectric material having a relative permittivity of 6 is used as insulation between the two conductors?

8. Refer to Fig. 7, page 360. Assume that $r = 0.5$ cm, $r_2 = 2.4$ cm, and that $\mathcal{E}_{2\,max.}$ is specified as being equal to $0.8\ \mathcal{E}_{1\,max.}$.

(*a*) Find the capacitance per meter length of bushing.

(*b*) If the conductors are at a potential difference of 100,000 volts, what is the potential drop across each layer of dielectric?

(*c*) What is the magnitude of $\mathcal{E}_{1\,\text{max.}}$ in part (*b*) expressed in kilovolts per meter?

9. Determine the wire-to-wire capacitance of a 20-mile parallel wire transmission line consisting of solid No. 00 AWG wire (diameter 0.365 in.) separated 4 ft center-to-center in air and well above the surface of the ground.

10. Determine the wire-to-wire capacitance (in micromicrofarads per meter) of two parallel No. 00 AWG conductors separated center-to-center in air by 4 ft if these conductors lie in a horizontal plane 2 ft above the ground plane.

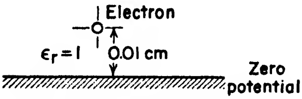


FIG. 20. See Prob. 11.

11. Calculate the force of attraction between the electron shown in Fig. 20 and the zero potential surface by the method of images.

12. Consider the upper and lower boundaries shown in Fig. 21 to be cross-sections of two long conducting plates separated by dielectric material having an ϵ_r of 6. What is the plate-to-plate capacitance if the plates have lengths of 13 m normal to the plane of the page?

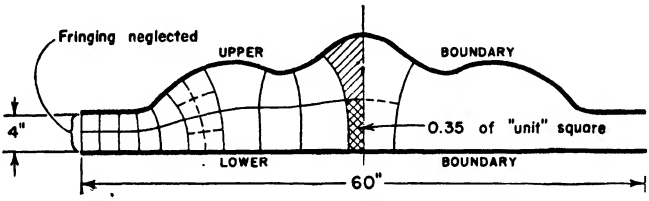


FIG. 21. See Prob. 12.

13. In Fig. 26, page 191, is shown the cross-section of two parallel plates. As indicated in this illustration, the plates are 2 in. wide and are separated by a distance of 1 in. The conductors are 100 in. long, and an air dielectric is assumed.

(*a*) Estimate, by means of a field map, the capacitance of the two plates.

Ans. $73\mu\mu$f.

(*b*) Compare the result found in (*a*) with that obtained by the parallel-plate formula which neglects fringing.

(*c*) Compare the result obtained in (*a*) with that obtained by the parallel-plate formula but allow for fringing to the extent of assuming that the plates have equivalent widths of 3 in. rather than the actual 2 in.

14. Show that the product RC is dimensionally equivalent to time t.

15. What is the time constant of a circuit consisting of a 0.5-megohm resistor in series with a 15,000-$\mu\mu$f capacitor?

16. A 10-μf condenser is charged initially to a voltage of 400 volts.

(*a*) What energy is stored in the condenser in its initial state of charge?

(*b*) What length of time is required for the stored energy to reach 10 per cent of its initial value if the discharge path is a 1-megohm resistor?

17. Refer to Fig. 14, page 374. The known data are

$$E = 1000 \text{ volts} \quad C = 4.0\mu\text{f} \quad R = 2 \text{ megohms} \quad Q_0 = 0.008 \text{ coulomb}$$

The polarity of Q_0 is that indicated in Fig. 14, and the switch S is closed at $t = 0$.

(*a*) What is the magnitude and direction of the circuit current just after switch S is closed? Express the result in microamperes.

(*b*) What are the values of stored energy at $t = 0$ and at $t = 20$ min?

18. Plot the charge-time and the current-time graphs of the charge and current variations of Prob. 17, evaluating points of q and i at $t = 0$, $t = 0.5t_c$, $t = t_c$, $t = 2t_c$, $t = 3t_c$, $t = 4t_c$, and $t = 5t_c$.

19. Refer to Fig. 22. The known data are

$$E = 1000 \text{ volts} \quad R = 2 \times 10^6 \text{ ohms}$$
$$C_1 = 4 \times 10^{-6} \text{ farad} \quad C_2 = 6 \times 10^{-6} \text{ farad}$$
$$Q_{01} = 0.0024 \text{ coulomb, the polarity being as indicated in Fig. 22}$$

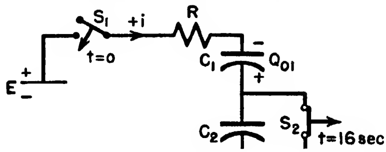


FIG. 22. See Probs. 19 and 20.

Switch S_1 is closed at $t = 0$ as shown, and switch S_2 remains closed for 16 sec after $t = 0$, at which time it is opened. Plot the current time variation from $t = 0$ to $t = 16$ sec.

20. What is the final or $t = \infty$ value of the voltage across condenser C_2 in Prob. 19?

21. A particular 0.1-μf condenser has a leakage resistance of 1 megohm. A sinusoidal a-c driving voltage

$$e = 10 \sin (50t) \quad \text{volts}$$

is applied to the terminals of the condenser.

(*a*) What is the frequency of the driving voltage in cycles per second? (In the above expression for e, t is expressed in seconds.)

(*b*) What is the maximum magnitude of the steady-state alternating current which flows through the generator e? Express the result in microamperes.

22. An audio-frequency voltage (of 796 cycles/sec) having a sinusoidal time variation and a maximum magnitude of 20 volts is applied to the terminals of a 4-μf condenser. What is the maximum magnitude of the steady-state alternating current through the condenser, the dielectric of which is assumed to be perfect?

23. A sinusoidal a-c driving voltage

$$e = 1414 \sin (377t) \quad \text{volts}$$

is applied to $R = 100$ ohms in series with $C = 26.53\ \mu$f.

(*a*) What is the impedance of the RC series circuit expressed in ohms?

(*b*) What is the effective value of the alternating current in the series circuit? Assume that the internal impedance of the source is negligibly small. (See page 105.)

(*c*) By what fractional part of a cycle does the current wave lead the applied voltage wave?

24. Where the center-to-center separation of parallel line wires D is less than about 8 times the radius of the conductors, the $+\sigma$ and the $-\sigma$ charges are attracted toward one another by a significant distance. This is indicated in Fig. 23 by $s/2 < D/2$. Under these conditions the wire-to-wire capacitance is

$$C_{\text{w-to-w}} = \frac{r}{36 \times 10^9 \ln\left[\dfrac{D}{2r} + \sqrt{\left(\dfrac{D}{2r}\right)^2 - 1}\right]} \quad \text{farads/m}$$

The problem is to prove, with the help of the major steps given below, that the above equation is true. Recognize at the outset that initially we do not know the distance s, even though we have replaced the actual surface charges by fictitious line charges at $x = s/2$ and at $x = -s/2$ as indicated in Fig. 23.

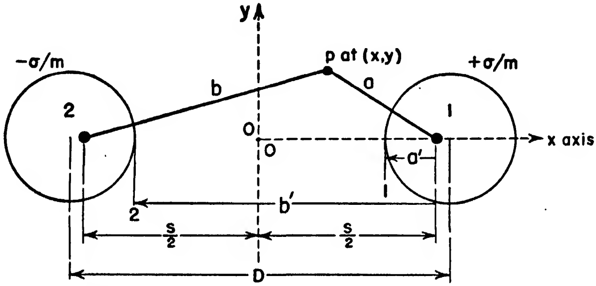


FIG. 23. For evaluation of capacitance of closely spaced parallel conductors. See Prob. 24.

(*a*) The potential difference between surface 1 and surface 2 due to both the $+\sigma$ and $-\sigma$ line charges is a constant which is equal to

$$V_{12} = 2\left[\frac{\sigma}{2\pi\epsilon_0\epsilon_r}\ln\frac{b'}{a'}\right] = \frac{(36 \times 10^9)\sigma}{\epsilon_r}\ln\frac{b'}{a'} \quad \text{volts}$$

where as yet b' and a' are undetermined distances. (See Fig. 23.)

(*b*) We should like to show that the condition $V_{12} =$ (a constant) is consistent with the fact that the equipotential surfaces of the conductors (in cross-section) are circles, the centers of which occupy the positions $x = D/2$ and $x = -D/2$. To this end, we write the absolute potential of *any* point in the plane, say point p in Fig. 23, as

$$V_p = \frac{(18 \times 10^9)\sigma}{\epsilon_r}\ln\frac{b}{a} \quad \text{volts} \quad (\text{due to both } +\sigma \text{ and } -\sigma)$$

(*c*) For V_p to be a constant, it is only necessary that $b/a = k$ where k is a constant yet to be determined. (The reader should satisfy himself that the above expression for V_p is correct and should then inquire into the physical significance of letting $b/a = k$.)

(*d*) From Fig. 23:

$$b = \sqrt{\left(\frac{s}{2} + x\right)^2 + y^2} \quad \text{and} \quad a = \sqrt{\left(\frac{s}{2} - x\right)^2 + y^2}$$

(*e*)
$$\left(\frac{s}{2} + x\right)^2 + y^2 = k^2\left[\left(\frac{s}{2} - x\right)^2 + y^2\right]$$

$$x^2 - \frac{(k^2 + 1)}{(k^2 - 1)} sx + \frac{(k^2 + 1)^2}{(k^2 - 1)^2}\frac{s^2}{4} + y^2 = -\frac{s^2}{4} + \frac{(k^2 + 1)^2}{(k^2 - 1)^2}\frac{s^2}{4}$$

or

$$\left[x - \frac{(k^2 + 1)}{(k^2 - 1)}\frac{s}{2}\right]^2 + y^2 = \frac{k^2 s^2}{(k^2 - 1)^2}$$

(*f*) Thus V_p = (a constant) can occupy the circle of radius,

$$r = \frac{ks}{k^2 - 1}$$

at the position

$$y = 0, \quad x = \frac{D}{2} = \frac{k^2 + 1}{k^2 - 1} \times \frac{s}{2}$$

where

$$k = \frac{D}{2r} + \sqrt{\left(\frac{D}{2r}\right)^2 - 1}$$

Thus

$$V_{12} = \frac{(36 \times 10^9)\sigma}{\epsilon_r} \ln k$$

CHAPTER XIV

Boundary Electromotive Forces (Faraday's Laws of Electrolysis)

1. Surface Layers. Conventional electromagnetic theory fails to take into account the fact that the surface layer of a material is different from the internal molecular layers of the material. Whereas the internal layers are bounded on either side by similar layers, the surface layer is exposed to the surface layer of a different material or possibly to free space.

Although the diagram of the molecular layers shown in Fig. 1 is only a schematic representation, it illustrates clearly enough why we may expect the outermost layer of a material to exhibit certain electrical effects—effects which are characteristic of particular molecular structures and which can be explained only in terms of surface phenomena peculiar to the material itself.

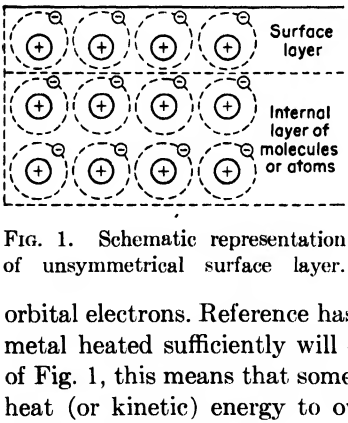


Fig. 1. Schematic representation of unsymmetrical surface layer.

The surface characteristic which is of immediate interest is the ability of the outermost layer of nuclei to hold their orbital electrons. Reference has previously been made to the fact that a metal heated sufficiently will emit electrons from its surface. In terms of Fig. 1, this means that some of the orbital electrons acquire sufficient heat (or kinetic) energy to overcome the forces which bind the electrons to the nuclei. Moreover, a strong electric field ($\mathcal{E}$) at the surface of a metal can extract electrons from the metal even at room temperatures; the rate of extraction depending upon the molecular structure of the material. Thus we know from experiment that different surface layers have different affinities for their orbital electrons, and it is in terms of *differences of these affinities* that many electrical effects can be explained.

2. Contact Potential Difference. Where two different metals are placed in contact with one another, a momentary transfer of electrons from one to the other will occur if the contacting surfaces are clean. The explanation is that the surface-layer nuclei of one material are capable of extracting some orbital electrons from the other. The transfer of electrons is only momentary because the metal which acquires electrons becomes negatively charged (and the other positively charged) with the result that a potential difference exists between the two metals, even though they are in contact.

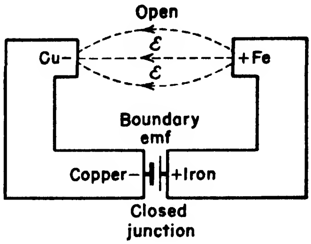


FIG. 2. Illustrating contact potential difference.

As applied to the iron-copper junction shown in Fig. 2, the iron loses electrons to the copper and becomes the positively charged metal. The contact junction between the two metals acts as though a small emf had been introduced at the boundary surface. In metals this boundary emf is usually referred to as contact potential difference.

Contact potential differences are usually only a fraction of a volt in magnitude and, except in special cases, do not make themselves felt in electrical circuits. If, for example, the opening between Fe and Cu in Fig. 2 were closed, another boundary emf (equal in magnitude and opposite in polarity to the one shown) would be introduced into the closed loop. No current would flow in the loop, and the two boundary emf's would remain concealed unless special methods were employed to detect their presence.

Contact potential differences can be measured in terms of the electric field established at the opening between the metals but only with difficulties which attend this indirect measurement. Contact potential differences do, however, manifest themselves in two different ways which we shall consider: the Peltier effect and the Seebeck effect.

3. The Peltier Effect. If current is caused to flow across the boundary between two dissimilar metals as shown in Fig. 3, it is found experimentally that heat is developed at one junction at a rate which is greater than the RI^2 rate. (See page 94.) The other junction actually absorbs heat from the surrounding air (or other medium), thus indicating that the surface is being cooled by the passage of current.

The heating at one boundary and the cooling at the other may be explained in terms of boundary emf's. If, as shown in Fig. 3, an abrupt potential drop is encountered at the iron-copper (Fe|Cu) boundary, the

positive charge in passing *from* iron *to* copper must lose energy. The energy lost by the charge in passing the Fe|Cu boundary is the result of the charge changing from potential E_1 to a lower potential E_2. This loss in energy appears as heat at the boundary in accordance with the relationship: heat energy $= (E_1 - E_2)Q$ joules.

The charge in passing *from* copper *to* iron (that is, passing the Cu|Fe boundary) is abruptly raised in potential if a boundary emf exists as indicated at the Cu|Fe junction in Fig. 3. The energy content of the charge Q is therefore raised as it passes *from* copper *to* iron, and this energy comes from the heat energy of the surrounding medium. The Cu|Fe boundary functions as a miniature refrigerator.

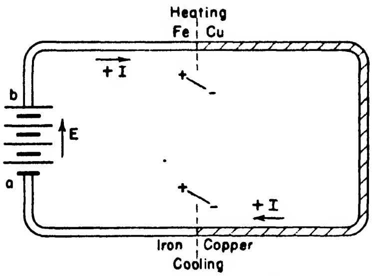


FIG. 3. The Peltier effect.

The boundary emf's are directed as shown in Fig. 3 because, if the direction of current is reversed, say by reversing the polarity of the battery, the two junctions interchange roles. With the current reversed, the upper junction shown in Fig. 3 becomes a Cu|Fe boundary and absorbs heat from the surrounding medium, thus indicating that charge is being raised in potential as it crosses this boundary.

The generation or absorption of heat energy at the boundaries of two dissimilar metals by the passage of current is called the Peltier effect. The voltage difference which appears at a boundary can be measured in terms of the heat energy developed or absorbed at the boundary per coulomb of charge passing the boundary and is called the Peltier emf. Thus

$$\text{Peltier emf} = \frac{W_b}{Q_b} \quad \text{volts} \tag{1}$$

where W_b is the number of joules of reversible heat energy developed or absorbed at the boundary in time Δt

$Q_b = I\,\Delta t$ is the number of coulombs passing the boundary in time Δt.

The Peltier emf is the net potential difference which exists at a current-carrying boundary, and the fact that it is lower in magnitude than the open-circuit contact potential difference indicates that the steady flow of electrons across a boundary lowers the contact potential difference from its open-circuit value.

The Peltier effect is of little or no importance from an engineering point of view because the heat energy (developed or absorbed) at a boundary is so very small that difficulty is experienced in measuring it accurately. The fact that the effect exists, however, is evidence that boundary emf's (or contact potential differences) exist at the junction of two dissimilar metals.

Example. It is found experimentally that 10 amp passing a Cu|Fe boundary for 100 sec absorbs 0.17 g-cal of heat from the surrounding medium.

Let it be required to evaluate the Peltier emf at the boundary. Since

$$(\text{No. of})\ \text{g-cal} = 0.2388EQ$$

$$\text{Peltier } E = \frac{0.17}{0.2388 \times 1000} = 0.000712 \quad \text{volt}$$

4. The Seebeck Effect. If the copper-iron junctions shown in Fig. 4 are at the same temperature, the boundary or junction emf's cancel when considered around the closed loop. If one traces the loop in the counterclockwise direction, for example, he finds that the voltage drop at the lower (Fe|Cu) boundary is canceled by the voltage rise at the upper (Cu|Fe) boundary.

If, however, one junction is heated as indicated in Fig. 4, a current will circulate in the closed circuit because a boundary emf is a function of the temperature of the boundary. Mildly heating the upper boundary in Fig. 4 while holding the other boundary at some lower temperature increases the boundary emf at the hot junction. The result is that a net voltage rise is encountered in tracing the loop in the counterclockwise direction. Current flows in the direction of net voltage rise, which in this case is from the cold junction to the hot junction in the copper as indicated in Fig. 4, provided the temperature is not greater than about 500°C. (See Prob. 3 at the close of the chapter.)

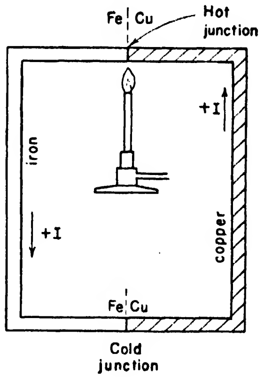
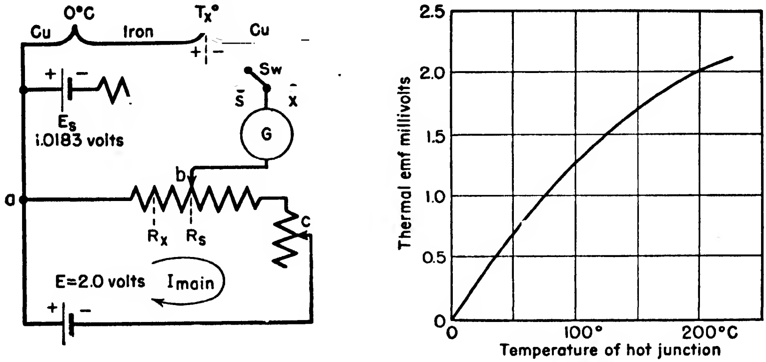


FIG. 4. The Seebeck effect.

The thermoelectric effect described above is often referred to as the Seebeck effect. The two principal uses of the Seebeck effect are: (1) measuring temperature (at one junction) in terms of the voltage developed in the circuit (see Fig. 5); (2) measuring high-frequency alternating currents in terms of the direct thermoelectric current (see Fig. 6).

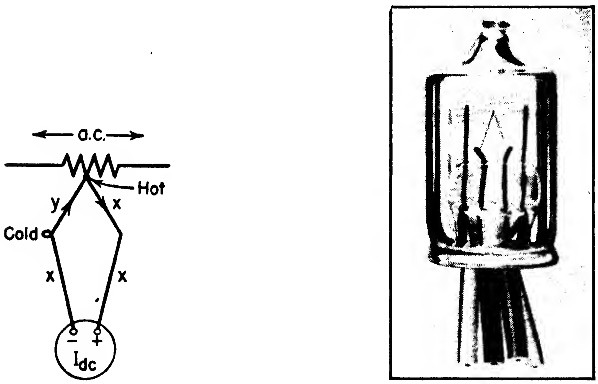
In measuring temperature, one junction is usually kept at a constant

temperature of 0°C and the other junction is placed at the point where the temperature is desired. The net voltage is then measured with some device which is capable of measuring millivolts, usually with a potentiom-



(a) (b)

Fig. 5. Thermocouple for measuring temperature T_x.



(a) (b)

Fig. 6. Thermocouple method of measuring high-frequency alternating current. In (b) is shown a vacuum thermoelement as used in Weston thermo-milliammeters and voltmeters. (Courtesy of Weston Electrical Instrument Corp.)

eter of the kind shown in Fig. 5-a. With the aid of a calibration chart, the voltage reading obtained by the Seebeck effect corresponds to a particular value (or values) of temperature.

In measuring high-frequency alternating currents, the hot junction is thermally connected to a resistor through which the high-frequency alternating current flows. The other junction (shown as the cold junction

in Fig 6) is usually left at room temperature, and the thermoelectric current which flows in the closed loop is read by means of a d-c milliammeter. Since the effective or ampere value of the alternating current is desired, the d-c milliammeter may be calibrated with direct current flowing through the heating resistor, thereby avoiding the necessity of a calibration chart.

Example. *Temperature Measurements.* The potentiometer shown in Fig. 5-a is a null-reading device in which the main circuit current I_{main} is *standardized* to a high degree of accuracy by means of the standard cell E_s. This standard cell has an open-circuit voltage which is known to about 1 part in 10,000. (See Section 9 of this chapter.)

If, as shown in Fig. 5-a, E_s has a known voltage of 1.0183 volts and I_{main} is to be *standardized* at 0.00100 amp, R_{ab} is set to the 1018.3-ohm position, marked R_s in Fig. 5-a. Then with switch Sw closed in the s position, I_{main} is standardized to the desired value by adjusting rheostat c until $I_s = 0$. This null reading is indicated by a sensitive galvanometer in the G position.

After I_{main} has been standardized as indicated above, any small voltage like the Seebeck voltage shown in Fig. 5-a may be balanced against a known $R_x I_{\text{main}}$ voltage drop simply by moving switch Sw to the x position and adjusting R_{ab} (to R_x) until again zero current flows through the galvanometer G.

If, for example, I_{main} in Fig. 5-a is first standardized to 0.0010 amp and then switch Sw is changed to the x position, it may be found that $R_{ab} = R_x$ must be adjusted to 2.000 ohms in order to make $I_x = 0$. By Ohm's law

$$E_x = R_x I_{\text{main}} = 2.000 \times 0.0010 = 0.0020 \quad \text{volt or 2 mv}$$

Transferring this value of voltage to the calibration chart shown in Fig. 5-b informs us that the temperature of the hot junction is 200°C.

5. Seebeck Voltage as a Function of Temperature. Seebeck or thermal voltages are widely used in the measurement of temperatures both in industry and in scientific research. Thermal emf's are independent of the lengths and cross-sectional areas of the wires used to make the thermocouples. The fact that thermocouples can be made of small wires (often No. 30 AWG) makes it possible to imbed the test junction in places that would be inaccessible to an ordinary mercury-in-glass thermometer. Various metal combinations like nickel-platinum and copper-advance are employed. (Advance is a nickel-copper alloy, the characteristics of which are shown in Table I, page 81.)

The graph of thermal emf versus temperature is, in general, nonlinear, as shown in Fig. 5-b. The graph is, however, a smooth curve which is either concave downward (as in Fig. 5-b) or concave upward depending upon the metal combination employed. Over limited ranges of temperature, the thermal emf may be represented quite accurately as

the sum of a linear term and a squared term as shown below.

$$E_{x-y} = aT + bT^2 \quad \text{microvolts} \tag{2}$$

where a and b are constants and T is temperature in centigrade degrees of the test junction, the other junction being held at 0°C. The subscript $x-y$ refers to the two metals used to form the thermocouple.

The constants a and b in equation (2) must be evaluated from experimental information. If, for example, a copper-advance couple is checked against a standard thermometer it is found to yield thermal voltages of

380 microvolts at 10°C temperature difference

2000 microvolts at 50°C temperature difference.

Hence

At 10°C, $\qquad E_{\text{Cu-ad}} = a(10) + b(100) = 380 \quad \mu v$

At 50°C, $\qquad E_{\text{Cu-ad}} = a(50) + b(2500) = 2000 \quad \mu v$

from which $a = 37.5$ and $b = 0.05$ for $E_{\text{Cu-ad}}$ expressed in microvolts. Thus

$$E_{\text{Cu-ad}} = 37.5T + 0.05T^2 \quad \mu v \tag{3}$$

Equation (3) fits the actual curve quite closely over the temperature range from 0°C to 100°C. For example, at 100°C the above equation yields

$$E_{\text{Cu-ad}} = 3750 + 500 = 4250 \quad \mu v$$

whereas an actual calibration point at 100°C yields 4200 μv.

For accurate work over wide ranges of temperature, it is desirable to have a large-scale calibration chart of the particular thermocouple which is in use since impurities in the metals may cause slight deviations in the voltage versus T graph.

6. Electrolytic Conductors. In order to appreciate the operating principle of a battery even in a general way, it is necessary first to consider the nature of the liquid or semi-liquid conductors which are used in batteries. Liquid conductors are usually referred to as electrolytic conductors because chemical reactions of some kind invariably attend the passage of current through liquid conductors.

A centimeter cube of distilled water will exhibit a resistance of approximately 5×10^5 ohms between opposite faces—about 3×10^{11} times the resistance of a corresponding cubic volume of copper. If, however, a small quantity of soluble acid, base, or salt is added to the distilled water, the resistance or resisitivity is greatly reduced. Sulphuric acid may, for example, be added until the specific gravity is about 1.2,

and the resistivity will be lowered to about 1.3 ohms/cm^3. Or potassium chloride may be added to reduce the resistivity of the distilled water from 5×10^5 ohms/cm^3 to about 10 ohms/cm^3.

The resistivity of a solution varies widely with the concentration of the solution and with temperature. In general, electrolytic conductors have negative temperature coefficients of resistivity; the higher the temperature, the lower the resistivity.

When a soluble material dissociates in the presence of water, electrically charged atoms (or groups of atoms) are formed. An example of this type of dissociation (for the case of a potassium chloride solution) is shown in Fig. 7. Potassium atoms lose electrons to the chlorine atoms during the dissociation process, and as a result the solution contains both positive and negative ions which act as charge carriers. If a potential

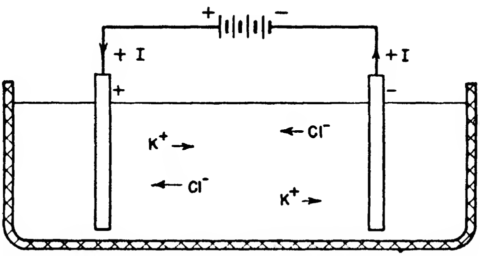


FIG. 7. Aqueous solution of potassium chloride containing K^+ ions and Cl^- ions.

difference is applied to the electrodes as shown in Fig. 7, the + carriers (K^+) move in the $+\mathcal{E}$ direction and the − carriers (Cl^-) in the $-\mathcal{E}$ direction.

The charge carriers being atomic in form cannot actually enter the metal electrodes but they do carry on the conduction process in the circuit where the electrons in the metallic portion of the circuit leave off, that is, at the boundary surfaces between the metal electrodes and the liquid portion of the circuit. An electron, upon arriving at the negative electrode of Fig. 7 on its journey from the negative terminal of the battery (or other emf source), is met by a potassium ion (K^+) and is taken into the solution by K^+. But for every electronic charge taken into the solution at the negative electrode, one is given back to the metallic portion of the circuit at the positive electrode by Cl^-. Thus a continuous flow of current is maintained in the circuit of which the electrolytic conductor is a part.

Although the resistance of an electrolytic conductor may be written as

$$R = \rho \frac{l}{A} \tag{4}$$

the resistivity ρ must be a space-averaged value along the l (or $\mathcal{E}$) direction to account for the variable ion concentration which exists in the electrolyte. In Fig. 7, for example, the potential gradient in the electrolyte produces concentration of K^+ ions near the negative electrode and a concentration of Cl^- ions near the positive electrode.

7. Elementary Primary Cells. Where two different metals are separated from one another by an electrolytic conductor, an effect similar to that of contact potential difference occurs. Electromotive forces are established at the boundary surfaces between the metal electrodes and the electrolyte.[1] If the metal electrodes are similar, no resultant emf is developed terminal-to-terminal because the boundary emf's cancel one another. But if the metals are dissimilar, a resultant emf is developed terminal-to-terminal as indicated in Fig. 8.

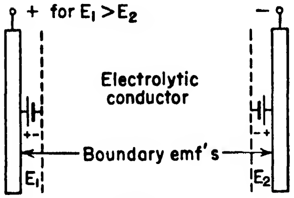


FIG. 8. Illustrating boundary emf's in an electrolytic cell.

In contrast to the contact potential difference between metal surfaces, the potential difference established by an electrolytic cell will deliver energy (in the form of EIt) to an external circuit. This energy comes

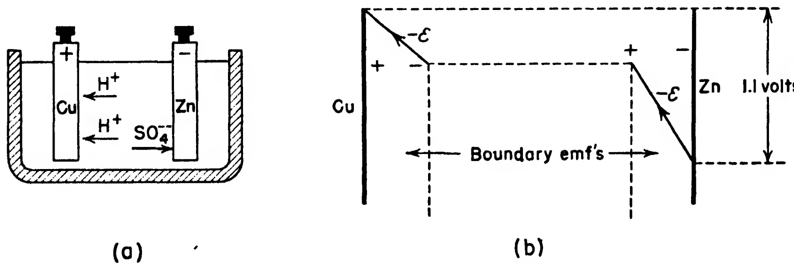


FIG. 9. A copper-zinc primary cell.

from the chemical decomposition of one or both of the metallic electrodes. If, for example, the electrolyte shown in Fig. 9 is a dilute solution of H_2SO_4, the positive ions are hydrogen nuclei (H^+) and the negative ions are doubly charged sulphate ions (SO_4^{--}). The H^+ ions travel to the copper electrode if an external circuit is connected to the terminals of the

[1] With certain types of electrolytes, the situation is somewhat less definite than indicated here. Boundary emf's may be established within the electrolytic conductor itself by rather sharply defined layers of the electrolytic conductor. In the early telegraph cells having copper and zinc electrodes and a sulphate solution, for example, a boundary emf existed between the layers of copper sulphate solution and zinc sulphate solution which divided rather sharply due to gravity. These cells were called "gravity" cells.

cell, and the SO_4^{--} ions travel to the zinc electrode under the influence of the potential gradients developed by the boundary emf's. As illustrated in Fig. 9, the plus ions are driven from $-$ to $+$ in the emf source developed at the copper electrode, and the negative ions are driven from $+$ to $-$ by the emf source developed at the zinc electrode. Essentially, the emf's within the cell result from the chemical affinities of the different boundary surfaces for one another.

The H^+ ions upon arriving at the copper electrode acquire an electron to form an atom of hydrogen, and two of these atoms in turn combine to form a molecule of diatomic hydrogen gas, H_2. If nothing is done to prevent the formation of hydrogen gas at the copper electrode, this electrode will become coated with hydrogen-gas bubbles, and the cell in this condition is reduced to a hydrogen-zinc cell effectively. The gas formation (called polarization) increases immensely the internal resistance of the cell which in turn destroys the usefulness of the cell as a current source.

The SO_4^{--} ions upon arriving at the zinc electrode combine with zinc to form zinc sulphate, $ZnSO_4$. It is this reaction which decomposes the zinc electrode. The number of coulombs that a cell can deliver depends upon the reserve amount of decomposable chemical material available within the cell. In a primary cell, the decomposed material cannot be reformed on the electrodes by the passage of current in the reverse direction as it can in a secondary or storage cell.

The Electromotive Force Series. The open-circuit voltage of a primary cell is dependent chiefly upon the metal electrodes which are used and only slightly upon the type of electrolyte which is used. The potential difference developed by the copper-zinc cell shown in Fig. 9, for example, is about 1.1 volts regardless of the electrolyte employed in the cell. The potential differences developed by the various metals operated in conjunction with a standard electrode[2] have been measured. An abbreviated list of these potential differences is given on page 398.

If two different metal electrodes are immersed in an electrolyte of uniform concentration, the potential difference established between the electrodes will be approximately the algebraic difference of the potentials in that list. The electrode which is the higher in the list will be the positive terminal of the cell.

Example. Let it be required to find the amount of zinc decomposed in the copper-zinc cell shown in Fig. 9 by the passage of 10,000 coulombs through the

[2] See *Handbook of Chemistry and Physics* for a description of the *standard* electrode which consists essentially of a hydrogen gas film which is formed on a platinum electrode.

cell. (This number of coulombs might, for example, be the result of 2 amp flowing through an external circuit for 5000 sec.)

Each coulomb of charge passing through the cell requires that 6.25×10^{18}, or $1/(1.6 \times 10^{-19})$, electronic charges be transferred through the electrolyte.

One zinc atom is decomposed by each SO_4^{--} ion which brings 2 electrons into the solution during the formation of $ZnSO_4$. Hence the number of zinc atoms dissolved by the passage of 10,000 coulombs is

$$\text{zinc atoms} = \frac{10^4(6.25 \times 10^{18})}{2} = 3.12 \times 10^{22}$$

The atomic weight of zinc (see Appendix B) is 65.38; meaning that 6.03×10^{23} atoms of zinc weigh 65.38 g. Hence the weight of a zinc atom is

$$\frac{\text{weight}}{\text{atom}} = \frac{65.38}{6.03 \times 10^{23}} = 10.85 \times 10^{-23} \quad \text{g}$$

and the weight of the dissolved zinc is

$$\text{weight of zinc} = (3.12 \times 10^{22})(10.85 \times 10^{-23}) = 3.39 \quad \text{g}$$

Problems of this kind may be solved without the detailed analysis given above by means of Faraday's laws which are stated in Section 11. The advantage of the present type of solution is that it shows in detail the decomposition of the zinc electrode atom by atom, and it is in this manner that the electrode is actually decomposed.

Platinum	Pt	+0.87	volt (relative to standard electrode)
Silver	Ag	+0.84	" " " " "
Mercury	Hg	+0.80	" " " " "
Copper	Cu	+0.34	" " " " "
Standard		0.00	
Lead	Pb	−0.14	" " " " "
Iron	Fe	−0.40	" " " " "
Zinc	Zn	−0.76	" " " " "
Manganese	Mn	−1.0	" " " " "

8. The Dry Cell. Most commercial primary cells are constructed as shown in Fig. 10 where, as indicated, the positive electrode is carbon and the negative electrode takes the form of a zinc container. This cell develops an open-circuit voltage of about 1.5 volts and is made in a variety of sizes. In the 2.5-in. by 6-in. size, a cell in good condition will deliver between 20 and 30 amp if the terminals are short-circuited through a low resistance ammeter. The internal resistance of the cell under these conditions is less than 0.075 ohm.

The term *dry* cell is a misnomer because moisture is essential to the cell's operation. The electrolyte takes the form of a moist paste containing ammonium chloride (NH_4Cl) and zinc chloride ($ZnCl_2$). In order to minimize the detrimental effects of polarization, the carbon

electrode is surrounded by a porous mass consisting of granular carbon and manganese dioxide, MnO_2. The manganese dioxide functions as an efficient depolarizer by absorbing the gas that would otherwise collect at the carbon electrode and seriously interfere with the operation of the cell as a current source.

Various chemical reactions occur when the cell is delivering current. One of the primary reactions is the dissolution of the zinc electrode by the Cl^- ions to form zinc chloride, $ZnCl_2$. The following chemical formula sums up the more basic transformations that occur when the cell is delivering current:

$$Zn + 2NH_4Cl + 2MnO_2 \rightarrow ZnCl_2 + 2NH_3 + Mn_2O_3 + H_2O \quad (5)$$

We may interpret the right-hand member of the above formula to mean that zinc chloride, ammonia gas, manganic oxide, and water are formed in the cell when it delivers current. These compounds are formed at the expense of the zinc, ammonium chloride, and the manganese dioxide which are present within the cell.

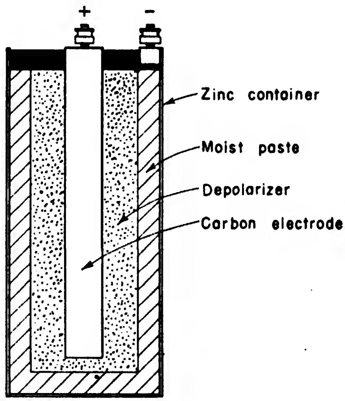


Fig. 10. Essential elements of an ordinary dry cell.

Some zinc is dissolved even when the cell is on open circuit. This type of dissolution results from *local action*, a name given to undesirable secondary reactions. Local action at the zinc electrode is caused principally by impurities in the zinc itself. These impurities, finding themselves in contact with the electrolyte, establish local emf cells at the surface of the zinc and cause deterioration of that surface. Owing principally to local action, an ordinary dry cell has a shelf life of only about two years.

9. The Weston Standard Cell. In Fig. 5-a is shown a cell marked E_s. This cell is used to standardize the current in the main circuit of the potentiometer, and this current in conjunction with the highly calibrated resistor R_{ab} can be used to measure unknown voltages and to calibrate electrical instruments to a high degree of accuracy. If it is assumed that R_{ab} can be calibrated and reliably read to one part in 10,000 (as it can in a good potentiometer), it becomes desirable to have a constant and reproducible source of emf which has the same order of accuracy as R_{ab}. Emf cells designed to have these characteristics are called *standard* cells.

Since a *constant voltage* is desired from a standard cell, *this type of cell is never used as a source of current* because polarization affects its terminal voltage.

The open-circuit voltage developed by the Weston standard cell shown in Fig. 11 possesses both constancy and reproducibility over relatively long periods of time. Two types of Weston standard cells are made: the *saturated* or *normal* cell and the *unsaturated* cell. The emf developed by a *normal* Weston cell at 20°C is so very close to 1.01830 volts and is so definitely reproducible that 1 volt has been legally defined in this country as 1/1.01830 part of the potential difference developed by this cell at 20°C. This cell consists of mercury and cadmium electrodes in an electrolyte of cadmium sulphate, and the entire cell is made

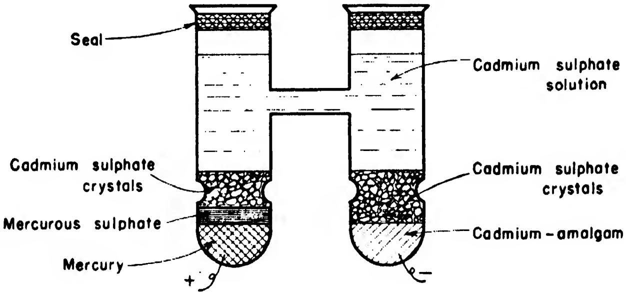


Fig. 11. Essential elements of a Weston standard cell.

according to a set of prescribed specifications. The U. S. National Bureau of Standards adopted the Weston normal cell as a standard of voltage in 1911.

The *unsaturated* cell is widely used in calibration laboratories because its emf is practically unaffected by temperature changes from 0°C to 40°C. Its open-circuit voltage may vary from cell to cell by a few parts in 10,000, but this variation is of no importance since the potential difference, whatever it may be, can be calibrated in terms of the emf of the *normal* cell. This calibration is usually performed at the Bureau of Standards, and each cell then carries with it a certificate of its open-circuit voltage to five or six significant figures.

10. Lead-Acid Storage Cells. The commonly used lead-acid storage cell has a positive electrode of lead peroxide (PbO_2) and a negative electrode of lead (Pb) immersed in a dilute solution of sulphuric acid (H_2SO_4). Porous insulators or separators are generally used between the plates even though these separators are not shown in Fig. 12.

Under normal conditions the cell develops a potential difference of

about 2 volts. This voltage depends to some extent upon the density or specific gravity of the acid electrolyte as shown below.

Concentration of Electrolyte		Emf
Specific Gravity	Per Cent by Weight of H_2SO_4	per Cell in Volts
1.050	7.31	1.903
1.100	14.33	1.956
1.150	20.91	2.000
1.200	27.32	2.045
1.250	33.33	2.091
1.300	39.19	2.138

On discharge, the H^+ ions in the electrolyte travel toward the positive PbO_2 plate, and the SO_4^{--} ions travel toward the negative Pb plate; and as a result lead sulphate is formed at each plate during the discharge period. The cell gradually loses voltage as the sulphation of the plates continues. If, however, the cell is subjected to a charging current (one which is opposite in direction to the discharge current), the lead

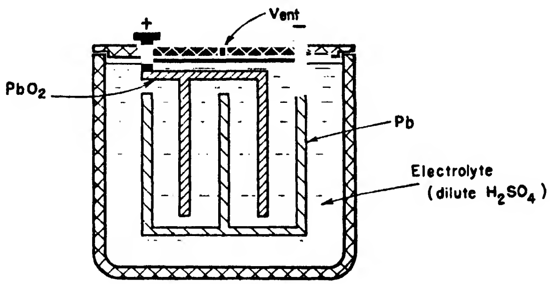


FIG. 12. Simplified diagram of the lead-acid storage cell.

sulphate ($PbSO_4$) is changed back into PbO_2 at the positive plate and back into Pb at the negative plate. Thus the cell is returned to its original state and is said to be charged and is ready for another discharge period.

The principal chemical reactions that occur at the electrodes of the lead-acid storage cell may be summed up in the following chemical equation which may be read from either left to right or from right to left:

$$\underset{+\text{plate}}{PbO_2} + \underset{\text{electrolyte}}{2H_2SO_4} + \underset{-\text{plate}}{Pb} \underset{\text{charge}}{\overset{\text{discharge}}{\rightleftarrows}} \underset{+\text{plate}}{PbSO_4} + \underset{\text{dilution}}{2H_2O} + \underset{-\text{plate}}{PbSO_4} \tag{6}$$

As indicated in the above equation, water is formed during discharge, and this water dilutes the electrolyte. The state of charge of the cell is

usually measured by means of a hydrometer which indicates the specific gravity of the electrolyte.

The capacity of a lead-acid cell is usually rated in ampere-hours, the ampere-hour rating depending largely upon the number and size of the plates. In testing a cell, it is customary to discharge the cell at a steady value of current which is equal to [(rated amp-hrs)/5] until the voltage of the cell drops to 1.7 volts. Thus the discharge period is approximately 5 hours in accordance with the automotive code which specifies the manner in which automobile batteries are to be tested.

The actual number of ampere-hours of charge delivered to the test circuit by the cell during the discharge period is arbitrarily selected as 100 per cent charge or 100 per cent output. The cell can deliver somewhat more than the arbitrarily selected 100 per cent charge, but to extend the discharge period beyond the point where the voltage per cell is less than 1.7 volts shortens the life of the cell.

Example. A 50-amp-hr lead-acid storage cell is assumed to be fully charged with an open-circuit voltage of 2.10 volts. Upon discharge at a 10-amp rate, it is found experimentally that:

(1) The terminal voltage drops abruptly to 2.07 volts at the start of the discharge period.

(2) The cell delivers 10 amp steadily for 4.8 hr before its terminal voltage drops to 1.7 volts.

(3) The time-averaged voltage during this test is 1.96 volts.

After the discharge test described above has been performed, the cell is charged at a 10-amp rate until the specific gravity of the electrolyte is brought up to its original value. This test shows that:

(1) 5.3 hr are required to bring the cell back to a full state of charge.

(2) The time-averaged voltage during this 5.3-hr period is 2.35 volts.

From these data we conclude that:

(1) internal resistance of the cell

$$R_{\text{int.}} = \frac{2.10 - 2.07}{10} = 0.003 \quad \text{ohm}$$

$$(2) \qquad \text{amp-hr efficiency} = \frac{10 \times 4.8}{10 \times 5.3} = 0.905 \quad (\text{or } 90.5 \text{ per cent})$$

$$(3) \qquad \text{watt-hr efficiency} = \frac{1.96 \times 10 \times 4.8}{2.35 \times 10 \times 5.3} = 0.755 \quad (\text{or } 75.5 \text{ per cent})$$

The values given above are comparable to those of the cells employed in automobile storage batteries.

11. Faraday's Laws of Electrolysis. The electrochemical reactions that occur at a boundary surface between a metallic conductor and an

electrolytic conductor were established about 1833 by Faraday. He found that an ion (in the electrolytic conductor) upon arriving at the metal electrode might be:

(1) deposited upon the electrode, or
(2) discharged at the electrode and then dissolved, or
(3) liberated from the electrolytic cell in the form of gas.

The mass M of the ions deposited, dissolved, or liberated is:

(1) proportional to the quantity of charge Q crossing the boundary;
(2) proportional to the mass (or ionic weight w) of the ion;
(3) inversely proportional to the number of electronic charges carried by the ion, that is, the valence of the ion, v.

These facts are known as Faraday's laws of electrolysis and may be summed up in the form of a single equation as

$$M = \frac{1}{\mathcal{F}} \frac{w}{v} Q_{(\text{coulombs})} \quad \text{g} \tag{7}$$

where $\mathcal{F}$ is a physical constant, numerically equal to the number of coulombs required to deposit or liberate w/v g (or 1 gram-equivalent) of any substance.

In the deposition of silver (having an atomic weight of 107.880 and a valence of 1), careful measurements show that 1 coulomb of charge deposits 0.0011180 g of silver. Hence

$$\mathcal{F} = \frac{1}{M} \frac{w}{v} Q = \frac{107.880}{0.0011180} = 96{,}494 \quad \text{coulombs/g} \quad \text{for silver} \tag{8}$$

$\mathcal{F}$ is actually a physical constant which represents the number of ions in w grams of the ionic substance (6.03×10^{23}) divided by the number of electronic charges per coulomb (6.25×10^{18}). Letting an electronic charge be symbolized by Q_e,

$$\mathcal{F} = \frac{1}{M} \frac{\dfrac{M}{\text{ion}} \times (6.03 \times 10^{23})}{\dfrac{Q_e}{\text{ion}}} Q = \frac{6.03 \times 10^{23}}{\dfrac{Q_e}{Q}} = \frac{6.03 \times 10^{23}}{6.25 \times 10^{18}}$$

or

$$\mathcal{F} = 96{,}500 \quad \text{(to three significant figures)} \tag{9}$$

Thus it may be seen that $\mathcal{F}$ is a constant which is far more universal in scope than equation (8) indicates.[3]

[3] The illusion is usually given in elementary texts that $\mathcal{F}$ is dimensionally equivalent to charge.

The relationship stated in equation (7) is not dependent upon the type of electrolyte or upon the type of electrodes employed. Neither is it dependent upon the rate at which the charge passes the boundary, although this time rate of charge (or current) may affect the "single-valueness" of the chemical reactions which occur at the boundary.

Example. The two electrolytic cells shown in Fig. 13 have the following constituents:

	Anode	Cathode	Electrolyte
Cell *a*	Silver	Platinum	Silver nitrate solution
Cell *b*	Copper	Copper	Copper sulphate solution

The electrode at which *current enters* a cell (or vacuum tube) is called the *anode*, and the electrode at which the *current leaves* is called the *cathode*.

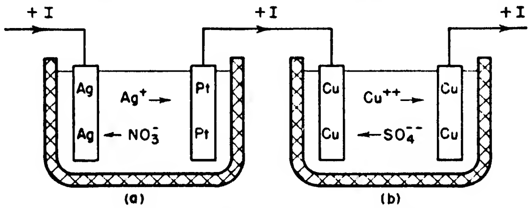


FIG. 13. Two electrolytic cells in series.

The primary reactions or chemical changes that occur at the four boundaries shown in Fig. 13 are listed below:

	Boundary	Chemical Changes
a	Ag | $AgNO_3$	$NO_3^- \rightarrow NO_3 \rightarrow AgNO_3$, where Ag is taken from the anode material. $AgNO_3$ dissociates: $AgNO_3 \rightarrow Ag^+ + NO_3^-$
	$AgNO_3$ | Pt	$Ag^+ \rightarrow Ag$ and is deposited upon the cathode in the form of silver plate'
b	Cu | $CuSO_4$	$SO_4^{--} \rightarrow SO_4 \rightarrow CuSO_4$ where Cu is taken from the anode material. $CuSO_4$ dissociates: $CuSO_4 \rightarrow Cu^{++} + SO_4^{--}$
	$CuSO_4$ | Cu	$Cu^{++} \rightarrow Cu$ and is deposited upon the cathode in the form of copper plate.

Let it be required to find the mass of the material which is deposited or dissolved at each of the four boundaries shown in Fig. 13 as a result of a steady current of 1 amp flowing through the cells for a period of 3 hr.

The quantity of electricity crossing each boundary is

$$Q = 1 \times 3 \times 3600 = 10{,}800 \quad \text{coulombs}$$

By equation (7), the mass of material at each boundary is

$$M = \frac{10{,}800}{96{,}500} \times \frac{w}{v} = 0.112\frac{w}{v} \quad \text{g}$$

w, the atomic weight of each element or group of elements, and v, the valence of each element or group of elements, may be found in Appendix B.

In determining the weight w of NO_3, for example, we add the atomic weight of N (14.0) to the atomic weight of 3O (48.0) to obtain 62.0. The valence v of NO_3 is 1 since it combines with silver, Ag, which has a valence of 1; or, electrically speaking, the valence v in equation (5) is the number of electronic charges carried by the ion which, in the case of NO_3^-, is 1 and, in the case of SO_4^{--}, is 2.

Thus

$$a\begin{cases}\text{The amount of nitrate, } NO_3, \text{ at anode is } (0.112 \times 62)/1 = 6.94 \text{ g.}\\ \text{The amount of Ag deposited on cathode is } (0.112 \times 107.88)/1 = 12.07 \text{ g.}\end{cases}$$

$$b\begin{cases}\text{The amount of sulphate, } SO_4, \text{ at anode is } (0.112 \times 96.06)/2 = 5.38 \text{ g.}\\ \text{The amount of Cu deposited on cathode is } (0.112 \times 63.57)/2 = 3.56 \text{ g.}\end{cases}$$

Many commerical processes make use of the electrolysis principle stated in equation (7). Metals are often refined by the plating process described in the above example. In this case the impure metal occupies the anode position, and a starting sheet on which the refined metal is plated is employed as the cathode.

Among the metals which can be electrolytically extracted from their ores are gold, silver, copper, zinc, nickel, cadmium, and aluminum. Other essential products which are electrolytically produced are hydrogen, oxygen, chlorine, calcium, sodium, potassium, lithium, barium, magnesium, and nitrogen. This list is far from complete and serves only to show the scope and commercial importance of the subject of electrochemistry.

12. Undesirable Electrolysis. Although the process known as electrolysis serves many useful purposes, it can under certain circumstances become a destructive agency. This situation sometimes arises along the ground return path of electrical circuits which are grounded at two or more points. If the circuit conditions are such that the grounded points (if left ungrounded) are at significant potential differences, earth currents will flow between these points when they are grounded. Earth currents, in selecting the path of least resistance between the grounded points, may enter and leave metallic conductors that are imbedded in moist earth. If direct current is involved, as in many traction systems, the earth currents may cause serious damage to water pipes and the like. The situation is represented schematically in Fig. 14.

The area (or boundary) where positive current enters the imbedded conductor in Fig. 14 is relatively unaffected by the current since this area functions as a cathode boundary on which positive ions (which are contained in $+\Delta I$) are deposited. The boundary at which ΔI leaves the imbedded conductor, however, functions as an anode boun-

dary. At this boundary the ions in the ΔI path attack the metal and dissolve it.

With alternating current, both boundaries are affected but the decomposition is only about 1 per cent of that at a d-c anode boundary

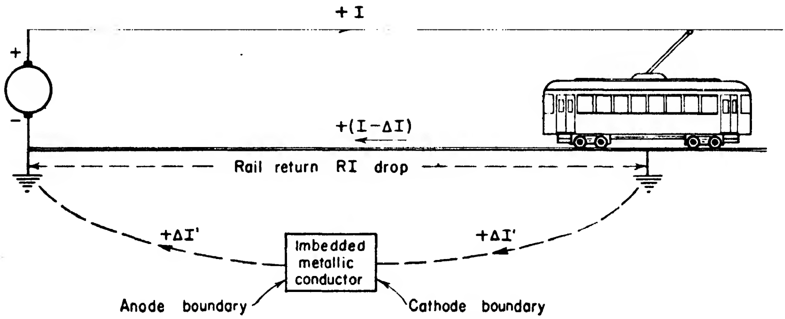


FIG. 14. An undesirable form of electrolysis.

under otherwise similar conditions, since metal dissolved on one-half cycle of the alternating current is redeposited (or replated) during the next half cycle.

PROBLEMS

1. The Peltier emf at the junction between two dissimilar metals is known to be 1 mv when 10 amp are crossing the junction or boundary surface.

What heat energy is generated (or absorbed) at the boundary in 5 min? Express the result in gram-calories.

2. For a copper-advance thermocouple it is known that $E_{\text{Cu-ad}} = 2000\ \mu\text{v}$ at 50°C and 4200 μv at 100°C.

(*a*) Write $E_{\text{Cu-ad}} = aT + bT^2\ \mu\text{v}$, where a and b are expressed numerically.

(*b*) How does the calculated value of $E_{\text{Cu-ad}}$ at 300°C (as obtained from the above equation) compare with the experimentally determined value of $E_{\text{Cu-ad}}$ which is known to be 14,800 μv at 300°C?

3. From the calibration curve shown in Fig. 5-b, page 392, we note that

$$E_{\text{Cu-Fe}} = 1.25 \text{ mv at } 100°\text{C} \quad E_{\text{Cu-Fe}} = 2.0 \text{ mv at } 200°\text{C}$$

(*a*) Write $E_{\text{Cu-Fe}} = aT + bT^2\ \mu\text{v}$ from the above data, expressing a and b numerically.

(*b*) At what temperature of the hot junction is the change of voltage with respect to temperature (dE/dT) equal to zero, as evaluated from the equation found in (*a*).

(*c*) From the equation found in (*a*), at what temperature of the hot junction (other than 0°C) is the net emf of the copper-iron thermocouple equal to zero?

(*d*) Compare the calculated values found in (*b*) and (*c*) with the corresponding values read from the actual voltage-temperature graph of the copper-iron couple shown in Fig. 15.

4. The thermal voltage developed by the hot junction of a nickel-lead thermocouple (with the cold junction held at 0°C) is

$$E_{\text{Ni-Pb}} = 18T + 0.018T^2 \quad \mu v$$

where T is in centigrade degrees. Positive current flows in the nickel from the cold junction to the hot junction if the circuit is closed. The corresponding equation for a platinum-lead thermocouple is

$$E_{\text{Pt-Pb}} = 3T + 0.013T^2 \quad \mu v$$

with positive current flowing through the platinum from cold junction to hot junction if the circuit is closed. What temperature is required at the hot junction of a nickel-platinum thermocouple to develop a voltage of 1.5 mv? Cold junction at 0°C.

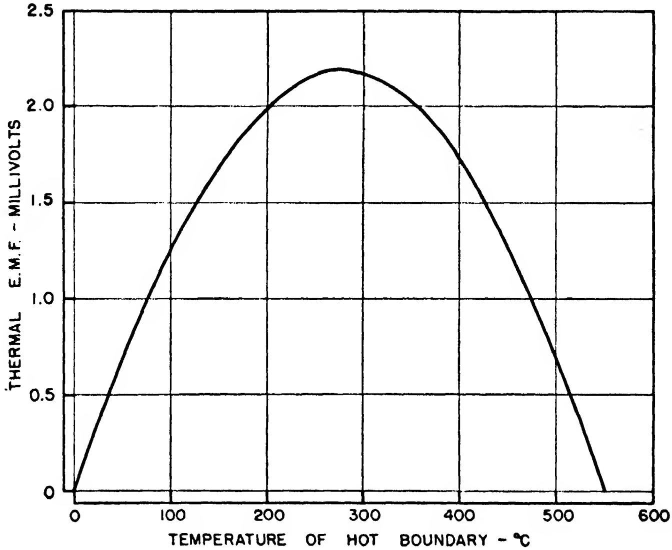


FIG. 15. Seebeck voltage developed by a copper-iron thermocouple. See Prob. 3.

5. What is the resistance of the electrolytic conductor shown in Fig. 12, page 401, if the metal plates are separated from one another by a distance of 0.20 cm and each active plate *face* has an effective area (presented to the electrolyte) of 100 sq cm? The resistivity of the electrolyte is 1.5 ohms/cm^3.

6. A saturated solution of sodium chloride (NaCl) is known to have resistivities expressed in ohms per centimeter cube:

$$\rho = 5.6 \text{ at } 10°\text{C} \qquad \rho = 4.4 \text{ at } 20°\text{C} \qquad \rho = 3.6 \text{ at } 30°\text{C}$$

What is the temperature coefficient of resistivity of this electrolyte at 20°C? Compare with the temperature coefficient of resistivity of copper at 20°C, which is 0.0039.

7. (*a*) What is the approximate potential difference developed by a primary cell consisting of silver and copper electrodes immersed in a dilute solution of hydrochloric acid, HCl? Which electrode is positive?

(*b*) Repeat (*a*) if the copper electrode is replaced with an iron electrode.

8. Find the mass of zinc dissolved from the zinc electrode of an ordinary dry cell during a 10-hr period in which the cell delivers a steady current of 2.0 amp.

9. What is the cost per kilowatt-hour (for the dissolved zinc alone) of the energy delivered by the dry cell in Prob. 8 if zinc costs $0.30 per lb and the time-averaged terminal voltage of the dry cell during the 10-hr period is 1.4 volts?

10. One type of radio battery consists of 45 small dry cells connected in series to give an open-circuit voltage of 67.5 volts. The useful life of these batteries when delivering a 10-ma load current is 15 hr. If the batteries cost $1.50 each, what is the cost per kilowatt-hour of the electrical energy generated by these batteries?

11. What mass of silver is deposited on the cathode (the Pt electrode) in Fig. 13-a by the passage of 10 amp for a period of 19,700 sec?

12. How many atoms of silver are deposited at the cathode in Prob. 11 during the 19,700-sec period? (Remember that in 107.88 g of silver there are 6.03×10^{23} atoms of silver.)

13. An electrolytic refining tank has an anode of impure copper, an electrolyte of acidified copper sulphate ($CuSO_4$), and a cathode of pure copper in the form of a starting sheet. The ions present in the electrolyte are Cu^{++} and SO_4^{--}. What cathode area must be employed if the current density at the cathode is not to exceed 0.03 amp/sq cm, if the copper is to be refined at the rate of 1 lb (or 453.6 g) per hr?

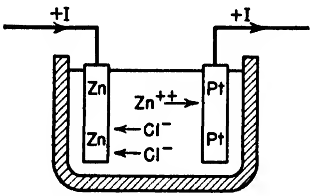


FIG. 16. See Probs. 14 and 15.

14. An electrolytic cell is formed with a zinc plate as anode, a platinum plate as cathode, and an electrolyte of zinc chloride, $ZnCl_2$, as shown in Fig. 16. A steady current of 2 amp is sent through the cell in the Zn-to-Pt direction for 13 hr and 24 min.

(*a*) If the reaction at the platinum cathode is simply

$$Zn^{++} \rightarrow Zn$$

find the number of grams of zinc delivered to the cathode.

(*b*) If the reaction at the zinc anode is simply

$$2Cl^- + Zn \rightarrow ZnCl_2$$

find the mass of zinc chloride ($ZnCl_2$) formed at the anode.

15. Secondary reactions frequently occur at the boundary surfaces between the electrolytic conductor and the electrodes of a cell. For the case considered in Prob. 14 (Fig. 16), zinc oxide (ZnO) and hydrogen gas (H_2) can be produced at the platinum electrode under the correct conditions of temperature, concentration of electrolyte, and current density. If the reactions at the platinum electrode are assumed to be entirely of the form

$$Zn^{++} + H_2O \rightarrow ZnO + 2H$$

find the mass of the zinc oxide (ZnO) formed and the mass of the hydrogen gas (H_2) liberated at the cathode boundary in Fig. 16 by a steady current of 2 amp flowing for 13 hr and 24 min.

16. What amount of lead is reduced to lead sulphate at the negative electrode of a lead-acid storage cell over the period of time in which the cell delivers 150 amp-hr?

17. A three-cell, lead-acid automobile storage battery has an open-circuit voltage of 6.30 volts. The internal resistance is 0.0020 ohm per cell.

(a) What is the terminal voltage of this battery when it is delivering 200 amp to the starter motor?

(b) Assuming that the internal resistance remains constant at the value 0.002 ohm per cell, what maximum horsepower can be delivered to an external load by this battery? (746 watts ≡ 1 hp.)

18. Two emf cells E_a and E_b having internal resistances of R_a and R_b respectively are connected to a load resistance R_L. Show that the load current is

$$I_L = \frac{E_aR_b + E_bR_a}{R_aR_b + R_aR_L + R_bR_L}$$

19. There are available N emf cells, each having an open-circuit voltage of e volts and an internal resistance of r ohms. It is desired to arrange them in the series-parallel combination (of n_s cells per series path in n_p parallel paths) which will deliver the greatest power to a specified load resistance R_L.

(a) Specify the number of parallel paths as a function of N, r, and R_L.

(b) Specify the number of cells in series per path, n_s, as a function of N, r, and R_L.

20. Show that the maximum power delivered to the load resistance in Prob. 19 is

$$P_{\text{max.}} = \frac{Ne^2}{4r} \quad \text{watts}$$

21. The number of cells available in Prob. 19 is 16. The open-circuit voltage of each cell is 2 volts, and the internal resistance of each cell is 0.10 ohm. The load resistance R_L is 0.40 ohm.

(a) What arrangement of the cells will give maximum load power?

(b) What is the maximum power that can be delivered to the load?

(c) What power will be delivered to the 0.4-ohm load resistor if the 16 cells are arranged in 4 parallel groups, each of these groups consisting of 4 cells in series?

CHAPTER XV

Non-Linear Circuit Elements

1. Classification of Resistive Circuit Elements. The current passing through some types of resistive circuit elements is not directly proportional to the voltage drop across the element. Under these conditions, the resistance ($R = V/I$) is variable, the graph of I versus V is not a straight line, and the resistance is said to be *non-linear*. Resistors which exhibit non-linear resistance are called *varistors* to distinguish them from the ordinary metallic and carbon resistors which, over reasonably small ranges of temperature, exhibit essentially constant resistances.

Varistors may be non-linear to the extent of passing current effectively in only one direction, in which case they are referred to as *unilateral* circuit elements. Thus variable resistors may be classified as:

(1) Symmetrical or bilateral varistors where current passes equally well in either circuit direction and where temperature is not a controlling factor.

(2) Negative-temperature-coefficient resistors or "thermistors" where small increases in temperature produce significantly large decreases in resistance.

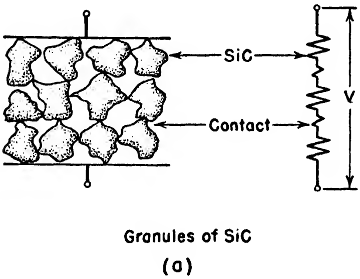
(3) Unilateral varistors where current passes effectively in only one direction.

The electrical characteristics of these resistors will be considered briefly in this chapter. Graphical methods will be used to solve circuit problems where one or more of these variable resistors are present in the circuit and, since these same methods are applicable to vacuum-tube circuits, the tube will be treated as a non-linear circuit element. In this connection, it should be noted that, whereas the vacuum tube is a non-linear unilateral *d-c* circuit element, it can under certain operating conditions be treated as a *linear* unilateral *a-c* circuit element. (See Sections 12 and 13.)

2. Symmetrical or Bilateral Varistors. A non-linear resistor which passes current equally well in either circuit direction is called a symmetrical or bilateral varistor. By non-linear is meant that the current-voltage

graph of the resistor, when plotted in rectangular coordinates, is not a straight line.

Thyrite is probably the most widely used of the symmetrical varistors. Thyrite consists of a large number of granules of silicon carbide (the familiar abrasive) bonded together by plastic clay. Each silicon carbide granule touches its neighbors in only a few small contact areas, as shown in Fig. 1-a. Since experiments show that the resistivity of silicon carbide itself is small and independent of voltage, the non-linear portion of Fig. 1-b does not originate in the granule itself. The fact that the resistance drops very markedly in the middle portion of Fig. 1-b is due to localized contact resistance between the different granules. This localized contact resistance is distinctly non-linear, decreasing sharply for increasing



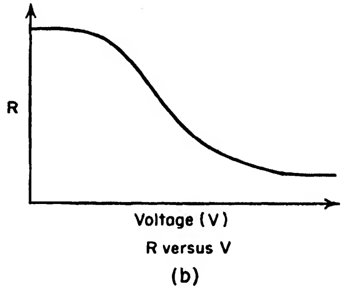


Fig. 1. Resistance of thyrite as a function of voltage.

values of potential gradient. The manner in which the overall resistance of a thyrite resistor varies with terminal voltage is shown in Fig. 2.

Thyrite resistors are used chiefly as protective devices in electrical circuits. They are usually made in the form of disks, the thicknesses of which range from about 0.02 in. to 1 in. At voltage gradients in the neighborhood of 100 volts/in., the thicker disks approach being insulators because under these conditions the current flow is only about 1 μ amp/sq in. of disk area. If, however, the voltage gradient is increased to 1000 volts/in., the current flow is about 0.1 amp/sq in. Thus a 10-fold increase in voltage results in a 100,000-fold increase in current, and it is this characteristic which is made use of where thyrite resistors are employed as protective devices.

Within the useful range of operation of thyrite resistors, the current-voltage relationship may be approximated quite closely by

$$I = (\pm)KV^n \tag{1}$$

where K and n are sensibly constant if the temperature is constant and the $(\pm)$ signs indicate that the current flows in either direction de-

pending upon the polarity of the applied voltage V. K depends largely upon the dimensions of the resistor, and n usually varies between about 4 and 5 depending upon the particular technique employed in manufacturing the thyrite.

The accuracy with which equation (1) defines the I-V relationship for any particular resistor may be determined by plotting I versus V

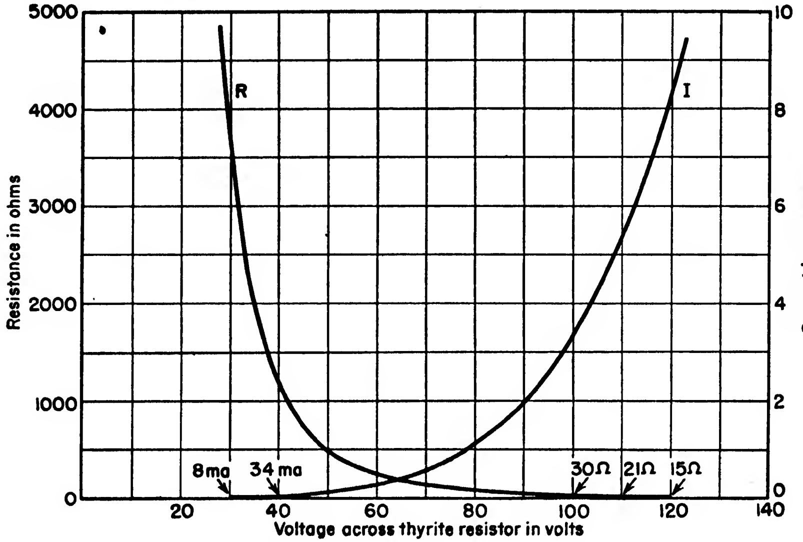


FIG. 2. Electrical characteristics of a thyrite resistor.

on log-log paper. If the resulting graph is reasonably straight over the operating range of the resistor, equation (1) is applicable, and the exponent n may be evaluated as the slope of the I versus V graph. This method of evaluating n follows directly from the straight-line equation of log I versus log V which is obtained by taking logarithms of both sides of equation (1). Thus

$$\log I = n \log V + \log K \tag{2}$$

and the value of n may be obtained directly as the slope of the I versus V graph on log-log paper. If, for example, the values of I shown in Fig. 2 at $V = 80$, 90, 100, 110, and 120 volts are plotted on log-log paper as shown in Fig. 3, the result is a straight line having a slope of 5. Hence n for this particular thyrite resistor is 5, and the value of K in equation (1) may be evaluated at some typical voltage point as

$$K = \frac{I}{V^5}\bigg]_{V=100} = \frac{3.33}{100^5} = 3.33 \times 10^{-10}$$

For the thyrite resistor having the I versus V characteristic shown in Fig. 2

$$I = (3.33 \times 10^{-10})V^5 \quad \text{amp} \tag{3}$$

where V is expressed in volts.

The manner in which a thyrite resistor may be employed to protect a circuit element against accidental voltage surges is indicated in Fig. 4. The circuit element R in Fig. 4 is protected in the sense that the thyrite resistor R_t, by virtue of its steep I versus V characteristic, prevents V from becoming excessively large even though the potential difference at points a and b is accidentally increased to 10 or 20 times its normal value. The details of this type of protection are given in the following example.

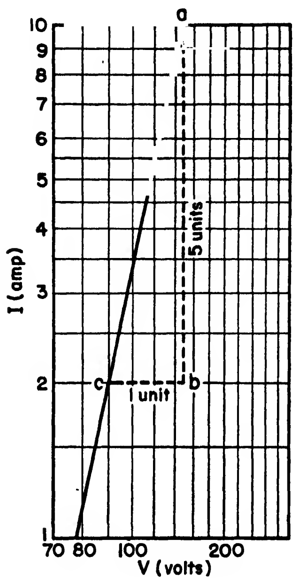


FIG. 3. Graphical evaluation for n in equation (1).

Example. It will be assumed that the thyrite resistor R_t in Fig. 4 has the I versus V characteristic shown in Fig. 2. Under normal operating

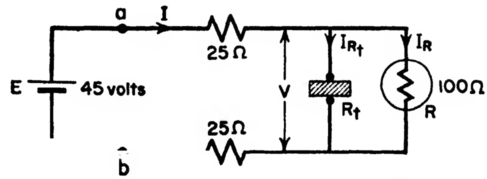


FIG. 4. A thyrite resistor R_t employed as a protective element.

conditions, the voltage across element R is essentially 30 volts, since R_t takes only 8 ma at 30 volts and R takes 300 ma at 30 volts.

If the voltage across points a and b in Fig. 4 were accidentally raised to 450 volts (owing, possibly, to cross-connections with another circuit), the voltage across element R would rise to 300 volts if the protective resistor R_t were not present. With R_t in the circuit, however, the voltage V is defined by Kirchhoff's voltage law to be

$$50I + V = 450 \quad \text{volts}$$

where

$$I = I_{R_t} + I_R = (3.33 \times 10^{-10})V^5 + \frac{V}{100}$$

The value of I_{R_t} in this particular case is obtained from equation (3). It

follows directly that

$$50\left[(3.33 \times 10^{-10})V^5 + \frac{V}{100}\right] + V = 450$$

from which

$$V \doteq 111 \text{ volts} \quad \text{(rather than 300 volts)}$$

In a similar manner it may be shown that voltage V in Fig. 4 rises to only about 137 volts if the potential difference between points a and b is raised to 1000 volts.

3. Graphical Solution of a Series Circuit. Many cases arise in engineering practice where a non-linear resistor is operated in series with a linear resistor across a fixed voltage. A common example is that shown in Fig. 5-a where the plate-to-cathode path of the vacuum tube is in series with a fixed resistor R_L. The non-linear I versus V characteristic

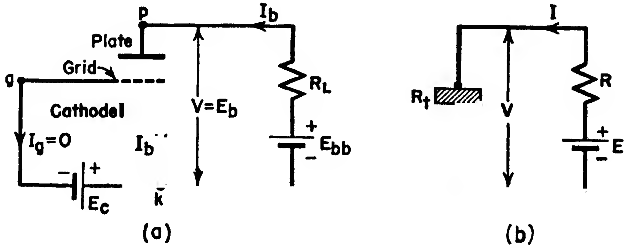


FIG. 5. A simple vacuum-tube circuit and the corresponding thyrite-resistor circuit.

of the tube is usually available in the form of a graph which is furnished by the manufacturer of the tube, and the problem is to find I when E and R are specified. For the present, a thyrite resistor will be used as the non-linear resistor in place of the plate-to-cathode path of the vacuum tube. (See Fig. 5-b.) Later the same type of graphical analysis as used to find I in Fig. 5-b will be used to find I in Fig. 5-a.

It is plain that in Fig. 5-b

$$RI + V = E \tag{4}$$

or

$$RI + R_tI = E \tag{5}$$

As it stands, equation (5) cannot be solved directly for I because R_t depends for its value upon the magnitude of I. Equation (4), however, may be used to effect a solution for I if the I versus V graph of the non-linear element is available. It will be assumed that this graph is available in the form of the curve shown in Fig. 6.

Since the origin of V is at the left end of the voltage axis in Fig. 6 and since we know from equation (4) that $RI = (E - V)$, it is a simple

matter to mark off E volts on the voltage axis and erect an I versus $(E - V)$ graph from this point as shown in Fig. 6. The current in the series circuit is then defined by the intersection of the I versus V graph

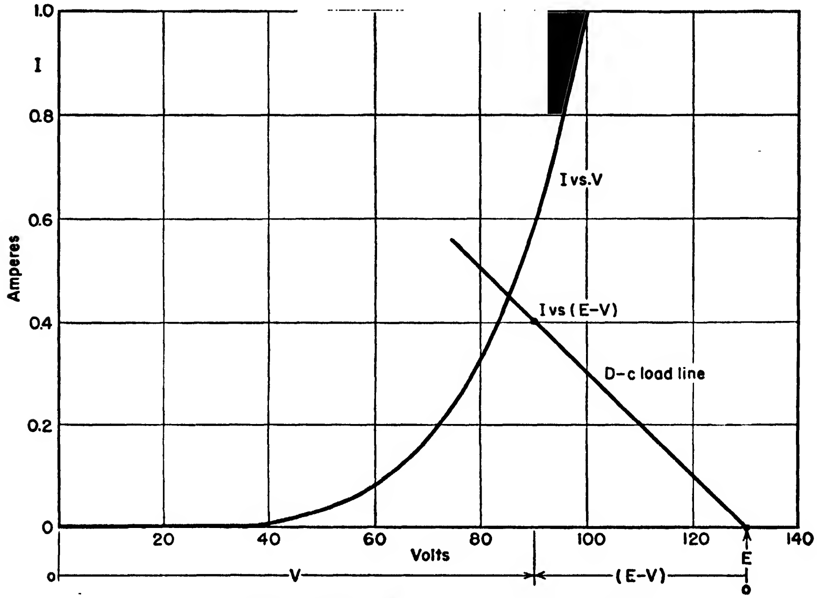


Fig. 6. See Example, Section 3. $I = 10^{-10} V^5$ amp.

and the I versus $(E - V)$ graph since the current I is common to both R and R_t. The details of a numerical case are outlined below.

Example. In the circuit arrangement shown in Fig. 5-b, let $E = 130$ volts, $R = 100$ ohms, and the I versus V characteristic be defined by the curve shown in Fig. 6.

(1) E is marked off on the voltage axis at the 130-volt point as shown in Fig. 6.

(2) The I versus $(E - V)$ characteristic is a straight line since R is assumed to be a linear resistor. Therefore, V may be assumed to have any value whatsoever (less than 130 volts) and a point on this straight line may be found. This point, together with point $(E, 0)$ defines the straight-line I versus $(E - V)$ graph. If, for example, we let $V = 90$ volts for the purpose of finding this one point,

$$I = \frac{130 - 90}{100} = 0.40 \quad \text{amp} \quad \text{[from equation (4)]}$$

(3) After drawing the straight line through the points (130, 0) and (90, 0.4), the current in the series circuit is found to be 0.45 amp at the intersection of the I versus V and the I versus $(E - V)$ graphs.

In vacuum-tube circuit analysis the I versus $(E - V)$ graph is called the d-c load line, and the I versus V graph is the plate current characteristic of the tube. If the simple graphical analysis employed here for the thyrite resistor is understood, no difficulty will be experienced in drawing d-c load lines for vacuum tubes.

If the linear resistor R in Fig. 5-b is replaced with a non-linear resistor, the graphical solution given above applies equally well provided the I versus V graph of this second non-linear resistor is substituted for the I versus $(E - V)$ graph shown in Fig. 6.

4. Negative-Temperature-Coefficient Resistors. Where the operation of a resistive circuit element is primarily dependent upon the temperature

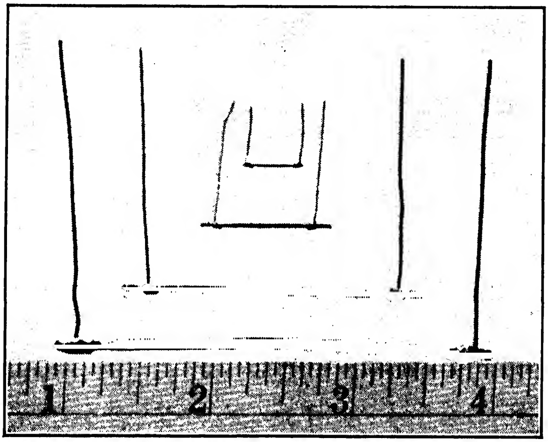


FIG. 7. One form of NTC resistor.

of the resistor, the element is called a thermal resistor or *thermistor*. Thermal resistors having relatively large negative temperature coefficients of resisitivity are sometimes sold under the name of "NTC" units. They are used extensively as temperature indicators and temperature compensators as well as in time-delay and other control devices.[1]

Many types of thermal resistors have been developed. Most of them are made from the oxides of heavy metals like iron, nickel, manganese, and uranium. These oxides are imbedded in a ceramic binder which often takes the form of plastic clay. The resulting resistor may take the form of a tiny bead mounted in an evacuated glass envelope, or it may take the form of the NTC units shown in Fig. 7.

[1] Small tungsten-filament lamps and small fuses are sometimes employed as positive-temperature-coefficient resistors to achieve the same results, but these units are not so sensitive to temperature changes as are the NTC units.

The manner in which the resistivity of a uranium oxide (U_3O_8) resistor varies with temperature is shown in Fig. 8. At 0°C, this resistor has a resistivity of 50,000 ohms/cm^3. (It should be noted that the ordinates of Fig. 8 are plotted to a logarithmic scale.) The resistivity decreases rapidly as the temperature rises, reaching 2800 ohms/cm^3 at 100°C, 420 ohms/cm^3 at 200°C, and 100 ohms/cm^3 at 300°C.

The temperature coefficient of resistivity of any material is defined as the per unit change in resistivity per degree change in temperature.

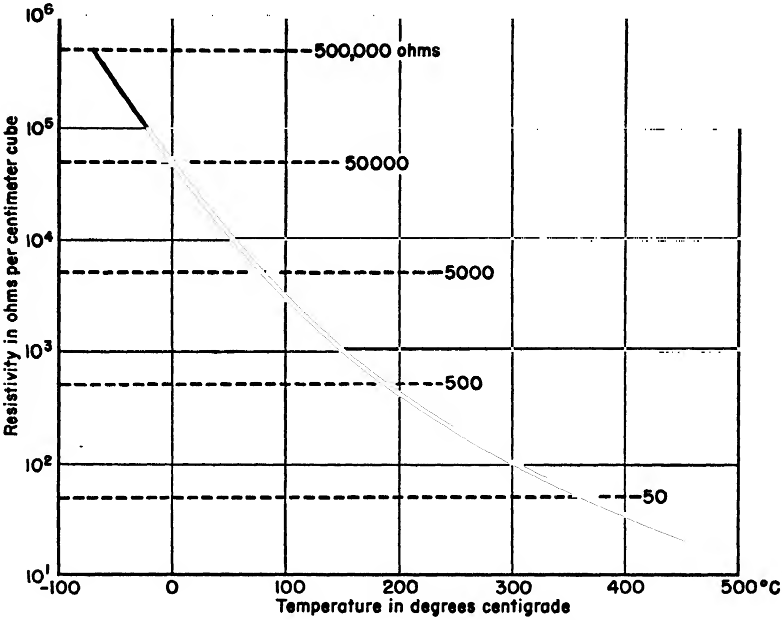


FIG. 8. Variation of resistivity of uranium oxide with temperature.

(See Chapter IV.) Applying this definition to a ±50°C interval about 0°C in the case of the uranium oxide resistor, we find

$$\alpha_0 = \frac{R_{+50^\circ} - R_{-50^\circ}}{R_{0^\circ}\,(T_{50^\circ} - T_{-50^\circ})} = \frac{10{,}200 - 250{,}000}{50{,}000(100)} = -0.048$$

Thus the temperature coefficient of resistivity of the oxide resistor is seen to be roughly ten times greater in magnitude than that of many pure metals, which is about +0.0042.

It has been found from experiment that the resistances of many NTC resistors obey the relationship

$$R_T = R_{303^\circ\mathrm{K}} \frac{C\varepsilon^{b/T}}{T^s} \tag{6}$$

where R_T is the resistance at $T°$ Kelvin (or $T°K$)
$R_{303°K}$ is the resistance at 303°K or 30°C
T is the temperature of the resistor in degrees Kelvin
C, b, and s are constants for any particular type of oxide resistor.

Two of the most widely used oxide resistors have the following constants:[2]

	C	b	s
Type 1	171,280	1306.5	2.8638
Type 2	141,600	1509.6	2.9477

Equation (6) is employed only where a high degree of accuracy is required since the expression is rather awkward to manipulate arithmetically. It is used, however, by some manufacturers of radiosondes, the small radio transmitters which are sent into the upper atmosphere by balloon to record weather conditions. In the following section, a formula for R_T is given which, although somewhat less accurate than equation (6), is much easier to manipulate.

5. Exponential Formula for Thermal Resistors. The following relationship is often employed in specifying the resistance of an NTC resistor at any temperature T in terms of the resistance at some other temperature T_0:

$$R_T = R_0 \varepsilon^{\beta[(1/T)-(1/T_0)]} = R_0 \frac{\varepsilon^{\beta/T}}{\varepsilon^{\beta/T_0}} \tag{7}$$

where R_T is the resistance at $T°$ Kelvin
R_0 is the resistance at some specified temperature T_0
T is the temperature in degrees Kelvin at which R_T is sought
β is a constant, over a limited range of temperature.

Equation (7) differs from the more exact form shown in equation (6) in that the variable T appears only in the exponent of equation (7), whereas it appears in both the exponent and in T^s in equation (6). The fact that T^s appears in the denominator of the more exact expression indicates that R_T actually decreases at a greater rate than that specified in equation (7).

To show that equation (7) is a reasonably accurate relationship over a limited temperature range, it is simply necessary to plot R_T versus the variable $1/T$ as shown in Fig. 9. Values of R_T and $1/T$ are obtained from the experimentally determined data shown in Fig. 8. It will be

[2] These data were furnished by Mr. C. B. Pear, formerly Director of Research at the Washington Institute of Technology.

observed that between 0°C and 150°C the semi-log plot of R_T versus $1/T$ is very close to a straight line. Also between 150°C and 400°C, the plot is essentially linear. It will be noted, however, that the slope of the

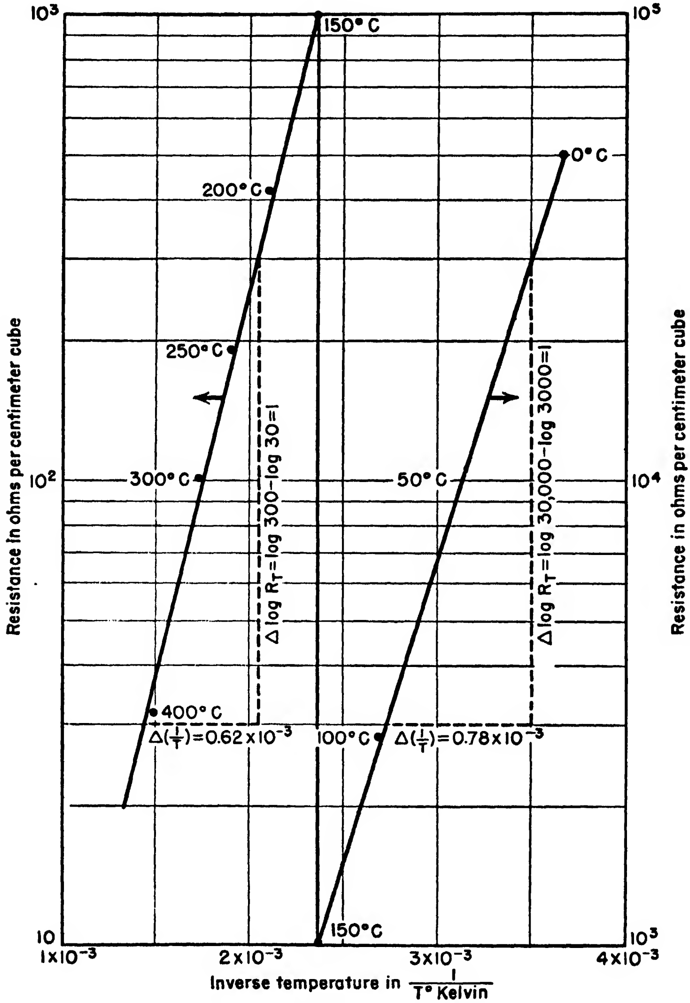


FIG. 9. Graphical method for evaluation of β.

line (from which β is determined) is greater at the higher temperatures, indicating that R_T actually decreases at a rate which is greater than that defined by $\varepsilon^{\beta/T}$.

If, in equation (7), it is recognized that $R_0/\varepsilon^{\beta/T_0}$ is a constant, say

K, it is plain that

$$\log R_T = (\beta \log \varepsilon)\left(\frac{1}{T}\right) + \log K$$

$$= (0.4343\,\beta)\left(\frac{1}{T}\right) + \log K \tag{8}$$

and that 0.4343β is the slope of the $\log R_T$ versus $1/T$ plot shown in Fig. 9. For the temperature interval between 0°C and 150°C,

$$\beta\Big]_0^{150} = \frac{\Delta \log R_T}{0.4343\,\Delta \dfrac{1}{T}} = \frac{1}{(0.4343)(0.78 \times 10^{-3})} = 2950$$

and for the temperature interval between 150°C and 400°C,

$$\beta\Big]_{150}^{400} = \frac{1}{(0.4343)(0.62 \times 10^{-3})} = 3640$$

Closer agreement between two successive β's may be had by selecting smaller temperature intervals, but for most practical work where thermal resistors are involved the temperature intervals selected above give the desired degree of accuracy.

Example. Assume that the resistance of a uranium oxide resistor is known to be 25,000 ohms at 0°C and that the value of β for this particular resistor is known to be 2950°K.

Let it be required to find the resistance at 100°C, employing equation (7).

$$R_{100°\text{C}} = R_{373°\text{K}} = 25{,}000\ \varepsilon^{2950(1/373 - 1/273)}$$

$$= 25{,}000\ \varepsilon^{-2.90} = \frac{25{,}000}{18.2}$$

$$= 1370 \quad \text{ohms}$$

The actual resistance of this resistor is known to be 1400 ohms from the original data presented in Fig. 8.

6. Temperature Measurements with Thermal Resistors. Since the resistance of a thermal resistor varies so widely with temperature, the resistor used in conjunction with a microammeter provides a simple means of measuring temperature. Either the series circuit shown in Fig. 10-a or the bridge circuit shown in Fig. 10-b may be employed for this purpose.

The objection to the series-circuit arrangement is the scale crowding that occurs near the lower part of the scale. This scale crowding can be overcome with the bridge type of circuit shown in Fig. 10-b. In either

type of circuit, care must be exercised in the design so that the current flowing through the NTC resistor will produce no appreciable self-heating within the resistor; otherwise the resistor is an unreliable temperature indicator.

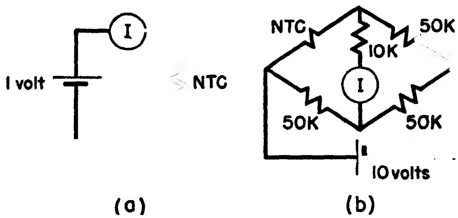


FIG. 10. Temperature indications using NTC resistors.

Example. It will be assumed that the NTC resistor shown in Fig. 10 has the following characteristics:

50,000 ohms (or 50K) at 0°C   10K at 50°C   2.5K at 100°C

Circuit calculations for the arrangements given in Fig. 10-a and Fig. 10-b will show:

Series Circuit (Fig. 10-a)	Bridge Circuit (Fig. 10-b)
I = 20 μamp at 0°C	I = 0 at 0°C
I = 100 μamp at 50°C	I = 77 μamp at 50°C
I = 400 μamp at 100°C	I = 121 μamp at 100°C

The advantage of the bridge type of circuit is clearly evident from these results.

7. Temperature Compensation. The effects of the positive temperature coefficient of resistivity of metallic conductors may be compensated for with negative-temperature-coefficient resistors. The general scheme is shown in Fig. 11 where the metallic resistors R_a and R_s are each assumed to have a resistance of 100 ohms at 20°C. As indicated in Fig. 11, these metallic resistors vary in resistance from 90 ohms at −20°C to 120 ohms at 100°C. In other words they have positive temperature coefficients (referred to −20°C) of

$$\alpha_{-20} = \frac{120 - 90}{90 \times 120} = +0.00278$$

By paralleling R_s in Fig. 11 with R_{NTC}, a resistance combination is obtained which has a negative temperature coefficient of the appropriate magnitude to make the overall resistance R essentially constant over the temperature range from −20°C to 100°C.

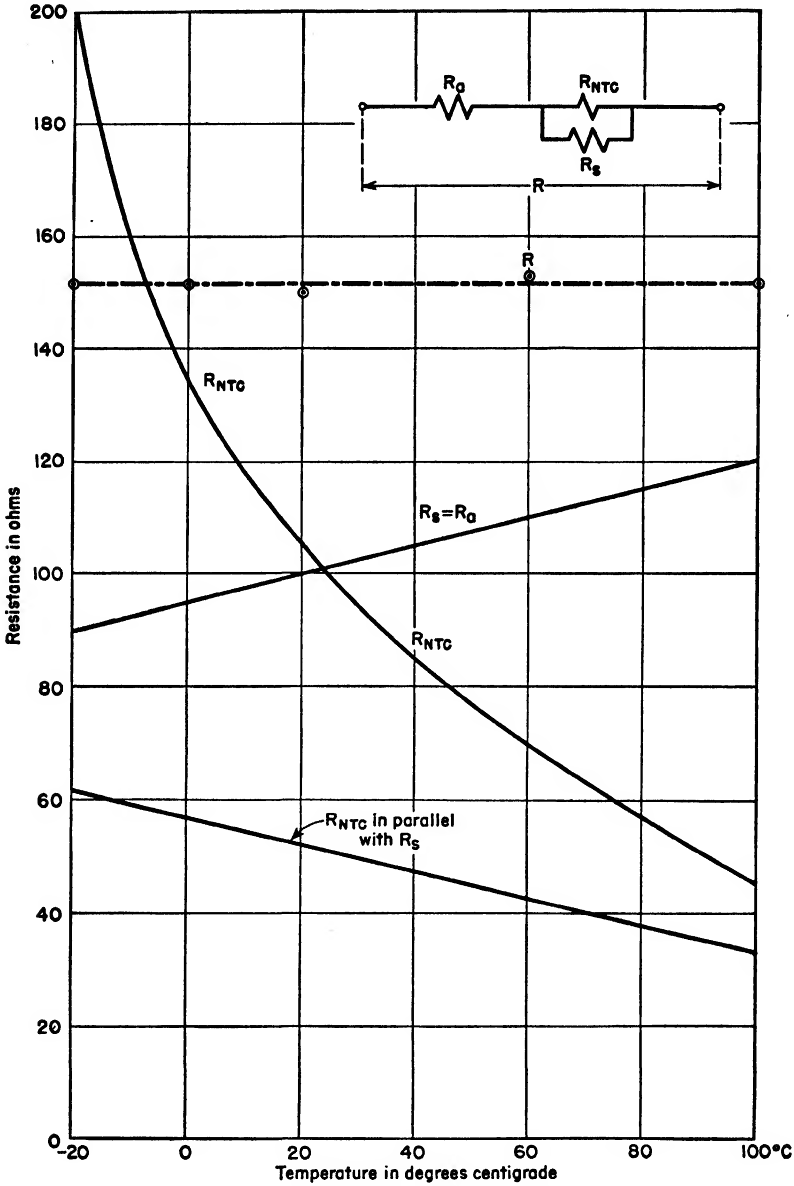


FIG. 11. Temperature compensation by means of an NTC resistor.

8. Other Uses of Thermal Resistors. Of the various practical applications of thermal resistors, only two will be considered here. The operation of a thermal resistor as a time-delay device is illustrated in Fig. 12. When switch S in Fig. 12-a is closed, the final or steady-state current through the load may be delayed for several seconds or even minutes by means of R_{NTC}. Initially the thermal resistor exhibits a very high resistance, and only a relatively small amount of current is allowed to pass through the load; but in time this small current heats the thermal resistor sufficiently to lower its resistance, thereby allowing larger values of load current to flow in the circuit.

In actual practice, the load shown in Fig. 12-a might be the winding of a relay, the contacts of which close a second circuit when the current through the relay winding reaches a value somewhat less than the final value of current shown in Fig. 12-b. In this way the second circuit cannot be completed until some specified time has elapsed after switch

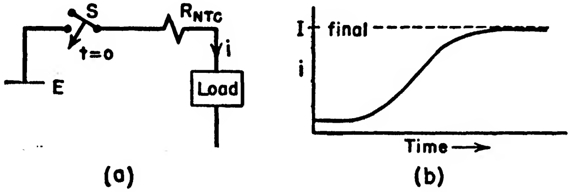


FIG. 12. A thermal resistor operating as a time-delay device.

S in Fig. 12-a is closed. If this switch is ganged with the main switch of the second circuit, the second circuit will not be actually completed until the relay in Fig. 12-a operates. Delay mechanisms of this kind are widely used in vacuum-tube circuits where a particular sequence of switching operations is necessary to prevent damage to the tubes.

Thermal resistors are symmetrical varistors which can be used equally well in either a-c or d-c circuits. The driving voltage E in Fig. 12-a might, for example, be an alternating emf, and the device would function in the same general manner as for the direct emf which is shown. The application shown in Fig. 12 operates on the principle of self-heating of the thermal resistor. In other applications the desired result is obtained by indirect heating of the thermal resistor as indicated in Fig. 13.

The a-c output voltage of the amplifier shown in Fig. 13, V_{out}, is used to heat the thermal resistor R_{NTC}, thereby reducing the magnitude of R_{NTC} when V_{out} reaches some preassigned value. A study of the input circuit of Fig. 13 will show that, for a fixed value of E_s, V_{in} will depend upon the magnitude of R_{NTC}. The lower the magnitude of R_{NTC}, the greater is the voltage drop in the R_1 resistor, and the smaller

the value of V_{in}. For low values of E_s and V_{out}, the heating effect is so small that R_{NTC} is very large, the result being that very little of the signal voltage E_s is lost in R_1. But when E_s becomes so large that it tends to overload the amplifier, V_{out} heats R_{NTC} to a point where a large

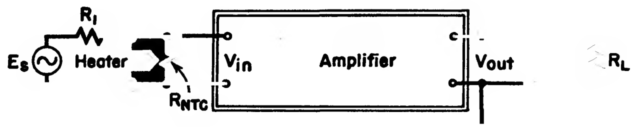


FIG. 13. A thermal resistor operating as an automatic control of the voltage which can be developed across R_L.

percentage of the signal voltage E_s is dissipated in R_1 and never reaches the input terminals of the amplifier. Hence R_{NTC} operating in conjunction with R_1 functions as an automatic control for the amplifier. If the amplifier shown in Fig. 13 is a speech amplifier, E_s is the output voltage of the microphone and R_L is the loudspeaker; and the feedback path from V_{out} to R_{NTC} will function as an automatic volume control of the loudspeaker output.

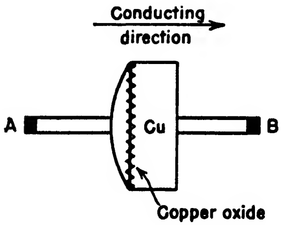


FIG. 14. Copper oxide rectifier.

9. The Copper Oxide Rectifier. The essential features of a copper oxide rectifier unit are shown in Fig. 14. This unit consists of a disk of sheet copper, one face of which has been so oxidized that a layer of red cuprous oxide is formed on this face. A layer of conducting material is sprayed or otherwise applied to the outer surface of this oxide, and this layer is known as the outer contact.

The boundary between the copper and the cuprous oxide exhibits a peculiar property which is akin to contact potential difference. (See Chapter XIV.) If a potential difference is applied between terminals A and B of the unit shown in Fig. 14, it is found that the current is not proportional to the applied voltage and that the polarity of the applied voltage has a marked effect upon the magnitude of the current. It is found that the current in the conducting direction (that is, from oxide to copper across the $CuO_2|Cu$ boundary) is from 100 to 1000 times greater than the current in the reverse direction for the same magnitude of applied voltage. This effect is illustrated in the I versus V graph of Fig. 15 where the reverse current is shown as the dashed line in the third quadrant. In reading Fig. 15, it should be noted that the scale of the

reverse current (that is, positive current across the Cu|CuO_2 boundary) is magnified 100 times relative to the scale of the forward current.

Experiments have shown that, as conductors of electricity, the body of the oxide and the outer contact obey Ohm's law and that the nonlinearity shown in Fig. 15 is due to the peculiar conductivity at the boundary between the copper and the copper oxide. This boundary

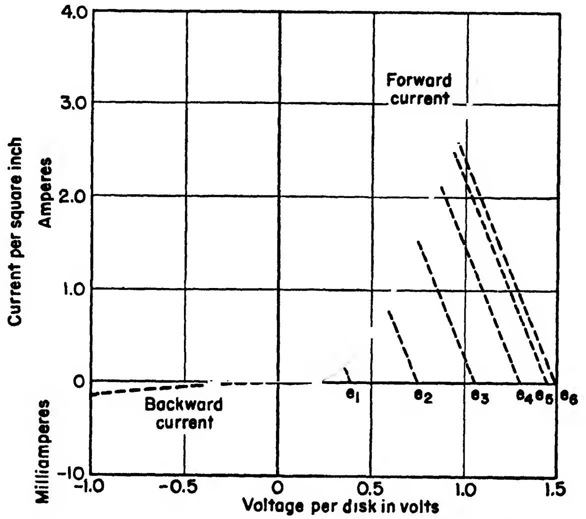


FIG. 15. I versus V graph of copper oxide rectifier.

functions to oppose the passage of current in the copper-to-copper oxide direction, that is, the passage of current from terminal B to terminal A in Fig. 14.

For most practical purposes, the copper oxide rectifier may be considered as a unilateral circuit element because the reverse current is so small relative to the forward current. The principal use of the copper oxide rectifier in elementary circuits is that of changing alternating current to direct current.

Example 1. *Simple Rectification.* If an alternating potential difference is used to energize the circuit shown in Fig. 16, it is plain that the current through the 0.2-ohm load resistor will be essentially in the BC direction because of the rectifying properties of the AB element.

Let it be required to find the time-averaged value of current flowing through the 0.2-ohm load resistor in Fig. 16 for the specified value of voltage

$$e = 1.5 \sin (377t) \quad \text{volts} \tag{9}$$

and for a rectifier disk area of 1 sq in.

Since the voltage applied to the series circuit is precisely defined at any time t by equation (9), and since the I versus V characteristic of the non-linear element is given in Fig. 15, the problem is essentially the same as that outlined in Section 3. The current flowing in the series circuit at any time t (after the beginning of a cycle of the applied voltage) may be found by the

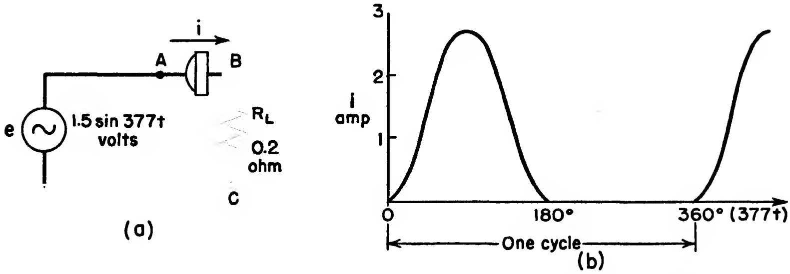


FIG. 16. A simple rectifier circuit and graph of rectified current.

graphical method shown in Fig. 6. After a series of these constructions is performed, the time-averaged value of current in the series circuit may be readily determined. The details are shown in tabular form below.

Time t, seconds	(377t) Radians	(377t) Degrees	e (volts) from equation (9)	i (amperes) from graphical constructions shown in Fig. 15
0	0	0	0	0
$\frac{1}{1440}$	0.2615	15	$e_1 = 0.39$	0.2
$\frac{1}{720}$	0.5230	30	$e_2 = 0.75$	0.8
$\frac{1}{480}$	0.7845	45	$e_3 = 1.06$	1.6
$\frac{1}{360}$	1.0460	60	$e_4 = 1.30$	2.2
$\frac{1}{288}$	1.309	75	$e_5 = 1.45$	2.6
$\frac{1}{240}$	1.570	90	$e_6 = 1.50$	2.7

The next quarter-cycle follows from symmetry, and the backward current (in the time interval $t = 1/120$ to $t = 1/60$ sec) is too small to be graphed in the i versus t plot shown in Fig. 16-b.

The time-averaged value of the current-time graph shown in Fig. 16-b, when considered over any integral number of cycles, is about 0.73 amp. It is this value of current that a permanent-magnet type of ammeter would register if placed in series with the circuit shown in Fig. 16-a.

Rectifiers for use at higher voltages are obtained by placing several copper oxide disks in series.

Example 2. *The Bridge-Type Rectifier.* Four copper oxide rectifying disks, arranged as shown in Fig. 17-a, may be inserted into an a-c circuit in such a way that the value of the alternating current flowing in the circuit can be

read on a d-c instrument. In studying the circuit arrangement shown in Fig. 17, it should be recognized that current can flow effectively only from the copper oxide *to* the copper; in Fig. 17-b this conducting direction is indicated by the arrow head which is associated with each of the four rectifying disks.

When terminal X in Fig. 17 is positive relative to terminal Y, positive current passes from X to Y by way of the $XWZY$ path which is through the d-c instrument in the $+$ to $-$ direction. During the following half-cycle of alternating current, terminal Y is positive relative to terminal X, and a positive current flows from Y to X by way of the $YWZX$ path which is also through the d-c instrument in the $+$ to $-$ direction, tending to make the instrument read up-scale.

By proper calibration, the d-c instrument may be made to read the effective value of the alternating current directly in amperes. This type of instrument is

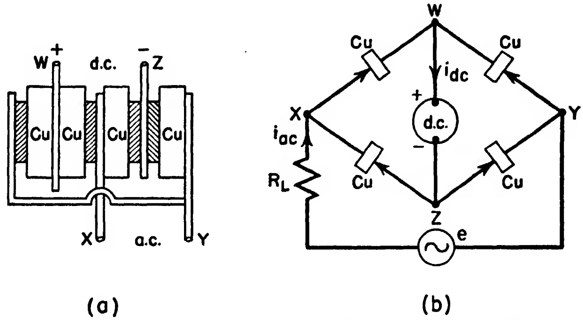


FIG. 17. A common method of measuring alternating current with a d-c instrument.

usually calibrated for sinusoidal current variations, and if the current variation which is being measured differs from a sinusoidal time variation the calibration will, in general, be in error. Since most of the alternating currents encountered in practice are approximately sinusoidal and since the arrangement shown in Fig. 17 is relatively inexpensive, this method of measuring alternating current is often used where a high degree of accuracy is not required.

10. The Physical Operation of a Diode Rectifier. A diode is a vacuum tube (or gas-filled tube) having two active electrodes, a plate and a cathode. The cathode of the ordinary diode must be heated; either directly as in the filamentary cathode shown in Fig. 18-a or indirectly as illustrated in Fig. 18-b. The heater voltages shown in Fig. 18 are often a-c voltages having effective values of either 2.5, 5, or 6.3 volts obtained from the secondary windings of small transformers which are energized from ordinary 115-volt, 60-cycle supply lines.

The diode has many uses in electronic circuits, all of which depend in

one way or another upon the rectifying property of the diode. A diode may be substituted for the copper oxide rectifier in Fig. 16-a, and the general performance of the circuit remains unchanged; the reason being that the diode passes positive current only from plate to cathode. The electrons emitted by the heated cathode of the diode are a source of negative charge carriers which travel in the cathode-to-plate direction within the tube when the plate is positive relative to the cathode. This flow of electrons is equivalent to positive current flowing from plate to cathode.

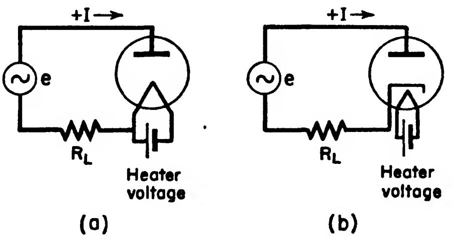


FIG. 18. Diode rectifiers.

When the plate is negative relative to the cathode, the electrons are repelled by the negative plate, and in effect the circuit is opened, thus preventing any appreciable amount of current from flowing in the circuit. The backward or reverse current in high-vacuum diodes is much less than the backward current of the copper oxide rectifier. (See Fig. 15.)

The current-voltage relationship of most high-vacuum diodes takes the same general form as given in equation (1), namely,

$$I = KV^n \quad (\text{for } +V \text{ only}) \tag{10}$$

provided the temperature of the cathode is sufficiently high to maintain a copious supply of electrons. The value of K depends upon the size of the electrodes, that is, upon the dimensions of the plate and the cathode. The exponent n in equation (10) varies from about 1.25 to 2 in practical diodes depending largely upon the configuration of the plate and the cathode. For concentric cylindrical electrodes, n has a theoretical value of 1.5.

If K and n in equation (10) are known or if the I-V characteristic of the diode is available, the problem of finding the rectified current in Fig. 18 differs in no essential respect from that given in Example 1, page 425.

Filtering networks consisting of series inductors and shunt capacitors are often employed to smooth out the rectified current shown in Fig. 16-b to a degree that, for most practical purposes, the rectified current is continuous and devoid of pulsations. The inductors are placed in series with R_L in Fig. 18, and the capacitors are placed in parallel with R_L to obtain the desired smoothing effect. The detailed analysis of the smoothing network is reserved for later courses since it logically follows or is a part of the a-c circuits course.

11. Triode Plate Characteristics and D-C Load Lines. The three-electrode vacuum tube shown in Fig. 5-a is called a triode. The triode differs from the diode by the grid which is inserted between the plate and the cathode. The grid is an open mesh, often a helically wound coil of wire. The electrons emitted by the cathode can pass through the grid provided the voltage of the grid relative to the cathode is not sufficiently negative to prevent the positive plate from attracting electrons to it.

The current which flows in the plate circuit of Fig. 5-a (I_b) is plainly dependent upon the voltage of the grid relative to the cathode and upon

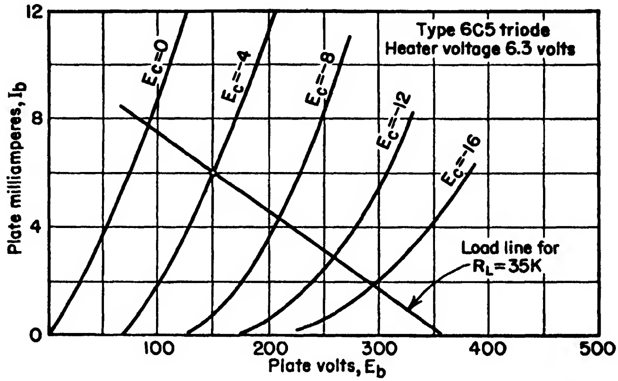


FIG. 19. Plate characteristics of a type 6C5 vacuum tube.

the voltage of the plate relative to the cathode. Thus for zero grid voltage ($E_{gk} = E_c = 0$ in Fig. 5-a), the plate current I_b versus E_b obeys equation (10) and takes the form of the $E_c = 0$ curve shown in Fig. 19. If, however, the grid voltage E_c is adjusted to some negative value, say -4 volts, the electric field established in the region between the grid and the cathode prevents the electrons from passing to the plate until the plate voltage reaches some finite positive value. For $E_c = -4$ volts in Fig. 19, the I_b versus E_b curve shows that the plate voltage E_b must be positive relative to the cathode by approximately 70 volts before any appreciable amount of plate current flows. For plate voltages higher than 70 volts, the plate current increases with E_b as shown by the $E_c = -4$ volts curve of Fig. 19.

If the grid is made still more negative relative to the cathode (say $E_c = -8$ volts), still higher plate voltages are required to obtain plate current. As shown in Fig. 19, no appreciable amount of plate current flows for $E_c = -8$ volts until the plate voltage is higher than 125 volts. For plate voltages higher than 125 volts, the plate current increases with E_b as shown by the $E_c = -8$ curve of Fig. 19.

The graphs of plate current I_b versus plate voltage E_b for various grid voltages E_c are called the plate characteristics of the tube, and it is this set of graphs which is usually employed in analyzing the circuit behavior of triodes. The first problem in triode analysis is usually that of determining the direct current which flows in the plate circuit when the plate supply voltage E_{bb}, the grid voltage E_c, and the d-c load resistance R_L are specified. (See Fig. 5-a.) The determination of the plate current under these specified conditions is essentially the same as that shown in Fig. 6.

Example. In Fig. 5-a, it will be assumed that the plate supply voltage is $E_{bb} = 360$ volts, the grid voltage is $E_c = -4$ volts, and the d-c load resistance is $R_L = 35{,}000$ ohms.

Let it be required to find the plate current I_b which flows in the circuit of Fig. 5-a, assuming that the triode is of the 6C5 type, the plate characteristics of which are shown in Fig. 19.

(1) E_{bb} is located at 360 volts on the E_b axis of Fig. 19.

(2) Some arbitrary voltage, say 160 volts, is laid off to the left of $E_{bb} = 360$ volts, and one point on the d-c load line is located as

$$I_b = \frac{160}{35{,}000} \doteq 0.0046 \text{ amp} \quad (\text{or } I_b = 4.6 \text{ ma})$$

at $E_b = 200$ volts.

This point ($E_b = 200$ volts, $I_b = 4.6$ ma) and the point ($E_b = 360$ volts, $I_b = 0$) determine the d-c load line as shown in Fig. 19.

(3) The intersection of the d-c load line with the I_b-E_b characteristic of the triode (for $E_c = -4$ volts) determines the current which flows in the series circuit of Fig. 5-a which includes E_{bb}, R_L, and the plate-to-cathode path of the triode.

(4) As shown in Fig. 19, the current in the series circuit is $I_{bo} = 6$ ma, and the voltage drop across the tube is $E_{bo} = 150$ volts.

The point E_{bo}, I_{bo} is called the d-c operating point of the triode.

12. The Triode Parameters: r_p, g_m, and μ. A study of the plate characteristics given in Fig. 19 will show that the plate current of a triode (i_b) is a function of both the plate voltage and of the grid voltage. Any change in plate current Δi_b is therefore dependent upon changes of plate voltage Δe_b and upon changes of grid voltage Δe_c. Thus

$$\Delta i_b = k_1\,\Delta e_b + k_2\,\Delta e_c \tag{11}$$

where k_1 is the change in i_b produced by a change in e_b only, $\left(\dfrac{\Delta i_b}{\Delta e_b}\right)$

k_2 is the change in i_b produced by a change in e_c only, $\left(\dfrac{\Delta i_b}{\Delta e_c}\right)$

In practice, the reciprocal of k_1, $\Delta e_b/\Delta i_b$, is usually employed to specify the variational plate resistance or internal a-c resistance of the triode. By definition, the a-c or variational plate resistance of a vacuum tube is

$$r_p = \left.\frac{\Delta e_b}{\Delta i_b}\right]_{e_c = \text{constant}} \qquad \left(\text{or } r_p = \frac{\partial e_b}{\partial i_b}\right) \tag{12}$$

The mutual conductance or transconductance between the plate (current) and the grid (voltage) is, by definition,

$$g_m = \left.\frac{\Delta i_b}{\Delta e_c}\right]_{e_b = \text{constant}} \qquad \left(\text{or } g_m = \frac{\partial i_b}{\partial e_c}\right) \tag{13}$$

If the plate voltage e_b is changed, the plate current can be brought back to its original value (thus making $\Delta i_b = 0$) by an appropriate change in grid voltage, Δe_c. Letting Δi_b in equation (11) equal zero, there is obtained for finite changes in e_b and e_c:

$$0 = \frac{1}{r_p}\,\Delta e_b + g_m\,\Delta e_c \tag{14}$$

or

$$-\frac{\Delta e_b}{\Delta e_c} = g_m r_p = \mu \tag{15}$$

where μ is called the amplification factor of the tube. The magnitude of μ specifies the ratio of plate voltage change (Δe_b) to grid voltage change (Δe_c) required to maintain constant plate current ($\Delta i_b = 0$), a positive increase in e_b requiring a decrease in e_c and vice versa as indicated by the minus sign in equation (15).

The tube parameters are usually determined directly from the plate characteristics of the triode as illustrated in Fig. 20-a. Thus for the operating point (E_{bo}, I_{bo}) in this figure,

$$g_m = \frac{\Delta i_b}{\Delta e_c} = \frac{(0.010 - 0.0024)}{[-2.0 - (-6)]} = 0.0019 \text{ mho} \tag{16}$$

$$r_p = \frac{\Delta e_b}{\Delta i_b} = \frac{(254 - 100)}{(0.016 - 0.001)} = 10{,}300 \text{ ohms} \tag{17}$$

$$\mu = -\frac{\Delta e_b}{\Delta e_c} = -\frac{(192 - 112)}{[-6 - (-2)]} = 20 \tag{18}$$

If the numerical values employed in the equations are properly identified on Fig. 20-a, the graphical method of determining g_m, r_p, and μ will be

self-evident. The value of g_m in equation (16), for example, is determined by measuring the current difference between $E_c = -2$ volts and $E_c = -6$ volts at constant plate voltage, $E_{bo} = 150$ volts. The smaller the increments of grid voltage, the more accurate will be the value of g_m. In practice, the mutual conductance of a tube is specified in micromhos.

The actual values of r_p and g_m depend upon the operating plate current I_{bo}, r_p decreasing as I_{bo} (or E_{bo}) increases and g_m increasing as

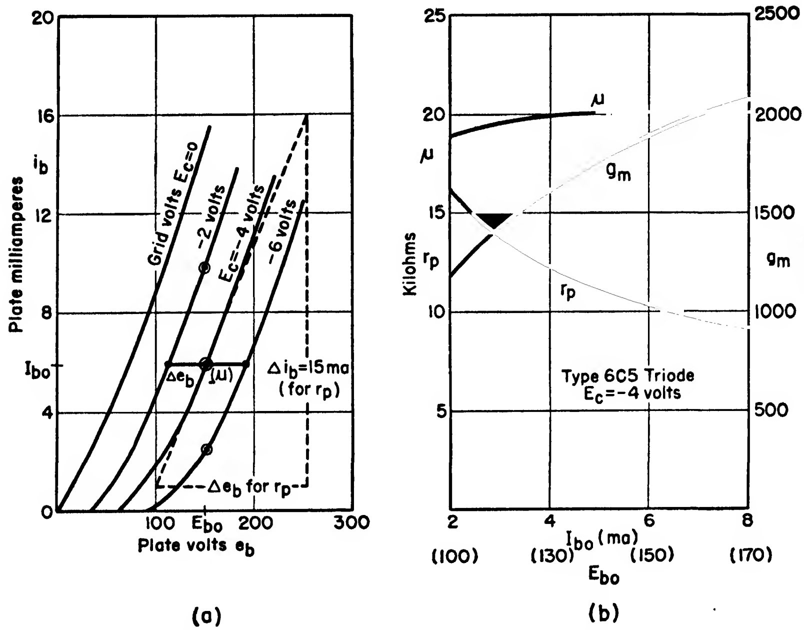


FIG. 20. Evaluation of g_m, r_p, and μ from the plate characteristics of the tube together with the variations of g_m, r_p, and μ relative to changes in I_{bo} and E_{bo}.

I_{bo} increases. These variations in r_p and g_m for the type 6C5 tube are illustrated in Fig. 20-b. It will be observed that $\mu = r_p g_m$ is reasonably constant over a wide range of plate current.

The variational (or a-c) resistance r_p is the internal resistance of the tube to the passage of small values of alternating current. The mutual conductance g_m specifies the degree to which a change in grid voltage can effect changes in plate current. The amplification factor μ is a measure of the ultimate voltage gain that can be obtained from the tube when it is used as an amplifier. (See Section 13.) Numerical values of r_p, g_m, and μ are usually specified by the manufacturer of the tube at two or three typical operating points. If, for any reason, these particular operating points cannot be used conveniently, the actual values of g_m

and r_p may be obtained directly from the plate characteristics as shown in equations (16) and (17).

Reference to Fig. 20-b will show the wide variations in g_m and r_p that occur when the d-c operating point is changed. Failure to recognize these wide variations may lead to serious errors in vacuum-tube circuit analysis.

13. The Equivalent Plate Circuit Theorem of a Triode. If a time-varying voltage is introduced into the *kg* branch of Fig. 5-a, this voltage will effect changes in the plate current Δi_b which flows through the load resistor R_L. Under normal operating conditions, the resulting variational (or a-c) voltage which appears across R_L is many times larger than the voltage introduced into the *kg* branch of the network. When

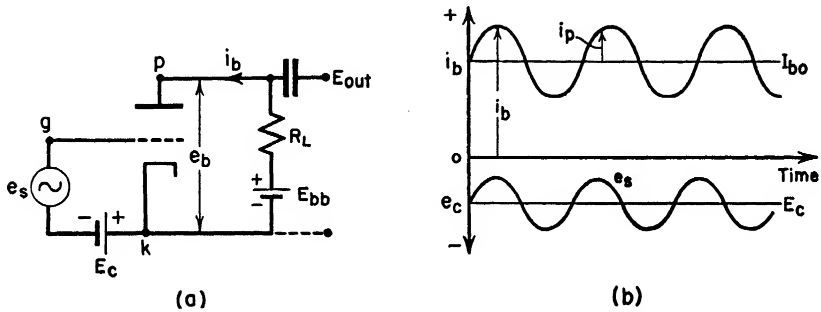


FIG. 21. A triode operating as a simple amplifier.

operated in this manner, the triode becomes a voltage amplifier, and as such it is used to amplify weak signal voltages to a point where the amplified voltage is sufficiently strong to drive loudspeakers, recorders, and other mechanisms.[3]

The great advantage of the triode as a voltage amplifier is that very little power is required from the voltage source which is undergoing amplification. Since the signal voltage is introduced into the *kg* path as shown in Fig. 21-a and since the grid *g* is maintained negative relative to the cathode *k*, no appreciable amount of current flows in the k-e_s-g path provided the a-c signal voltage e_s does not have a maximum magnitude which exceeds the bias voltage E_c. (See Fig. 21-b.) Under these conditions, the grid is never positive relative to the cathode, and no appreciable amount of current flows in the k-e_s-g branch of Fig. 21-a. The voltage source e_s is therefore required to furnish only an

[3] The triode is also widely used in current amplifiers, power amplifiers, oscillators, modulators, demodulators, and in various other ways. The explanation of these devices constitutes an essential part of a course in engineering electronics.

inappreciable amount of power to the k-e_s-g branch since $e_s i_g$ is negligibly small if i_g is negligibly small.

The variational grid voltage e_s causes a variational component of plate current i_p to flow in the plate circuit. (See Fig. 21-b.) The variational component of plate current is superimposed on the d-c component I_{bo} which is defined in magnitude by E_{bb}, E_c, and R_L as shown in Fig. 19. Thus the total plate current flowing when a signal voltage e_s is present is

$$i_b = I_{bo} + i_p \tag{19}$$

where I_{bo} is the steady operating current and i_p is the variational or a-c component of the total plate current i_b.

Since I_{bo} is essentially constant, the *change* in i_b is equal to i_p and, from equation (11),

$$\Delta i_b = i_p = \frac{1}{r_p}\,\Delta e_b + g_m\,\Delta e_c \tag{20}$$

The total voltage from plate to cathode of the triode is

$$e_b = E_{bb} - R_L i_b \tag{21}$$

where E_{bb} is the constant plate supply voltage as shown in Fig. 21-a. Hence

$$\Delta e_b = -R_L i_p \tag{22}$$

and, from equation (20),

$$i_p = \frac{-R_L i_p}{r_p} + g_m e_s \tag{23}$$

where e_s is the *change* in grid voltage Δe_c, and g_m is the mutual conductance of the triode. Thus

$$i_p = \frac{g_m r_p e_s}{R_L + r_p} = \frac{\mu e_s}{R_L + r_p} \tag{24}$$

which is called the equivalent plate circuit theorem of a triode. In words, this theorem states that the triode may be replaced by a voltage source equal to μe_s in series with the a-c resistance of the triode r_p, the assumption being that only the a-c operation of the tube is of importance. The a-c circuit diagram of the triode is shown in Fig. 22 where the pk path of the actual tube is replaced by a voltage source μe_s in series with the a-c resistance of the tube, r_p.

Either instantaneous, effective, or maximum values of current and voltage may be used in equation (24). Where sinusoidal variations of e_s are encountered, it is customary to write equation (24) as

$$I_p = \frac{\mu E_s}{R_L + r_p} \tag{24-a}$$

where E_s and I_p are the effective values of the signal voltage and a-c plate current respectively.

The output voltage in Fig. 21-a is

$$E_{\text{out}} = -R_L I_p \tag{25}$$

where the minus sign accounts for the physical fact that, when i_p increases, e_b decreases owing to the increased voltage drop across R_L. A study of the relative polarities given in Fig. 22 will show that, when E_s and E_{out} are both measured relative to the common cathode, point k, E_{out} goes down in potential when E_s goes up in potential and vice versa.

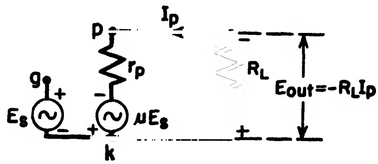


FIG. 22. Equivalent a-c circuit of a triode.

From equation (24-a) it follows that

$$E_{\text{out}} = \frac{-\mu E_s R_L}{R_L + r_p} \tag{26}$$

The voltage gain of the triode is the ratio of the output voltage E_{out} to the signal voltage E_s; or

$$\text{voltage gain } A = \frac{E_{\text{out}}}{E_s} = -\frac{\mu R_L}{R_L + r_p} \tag{27}$$

It will be observed that the voltage gain approaches the amplification factor of the tube μ as an upper limit when R_L is very much greater than r_p.

The above analysis is valid only when the triode is operated with a negative grid voltage and with E_{bo} and I_{bo} of such values that the plate characteristics (i_b versus e_b) are reasonably straight and parallel in the vicinity of the operating point (E_{bo}, I_{bo}). Furthermore the maximum magnitude of the signal voltage must be sufficiently small to keep the change in grid voltage within the limits where the i_b versus e_b curves are equally spaced. If these conditions are not met, the triode operates as a non-linear circuit element, and special graphical methods must be employed in order to analyze and predict the circuit performance of the tube. These special graphical methods will not be considered here.

Example. Let it be assumed that the triode shown in Fig. 21-a is a 6C5 which is operated as follows:

$E_c = -4$ volts $\quad E_{bb} = 360$ volts $\quad R_L = 35{,}000$ ohms $\quad e_s = 4 \sin (5000t)$ volts

It is required to find the voltage gain of the triode operating under these conditions.

For the specified value of e_s, the grid voltage swings between 0 and -8 volts since

$$e_c = E_c + e_s = -4 + 4 \sin (5000t) \quad \text{volts}$$

The i_b-e_b curves for this condition of operation are reasonably straight and parallel as shown in Fig. 19; hence equation (27) is applicable in finding the voltage gain of the amplifier.

From the calculations given in Section 12 (or from Fig. 20) we note that for the operating point $E_{bo} = 150$ volts, $I_{bo} = 6$ ma:

$$r_p = 10{,}300 \text{ ohms} \qquad \mu = 20$$

Hence the voltage gain of the triode is

$$A = \frac{-\mu R_L}{R_L + r_p} = -\frac{20 \times 35}{35 + 10.3} = -15.5$$

The value 15.5 for the voltage gain may be interpreted physically by referring to Fig. 19 and noting that the grid voltage varies between 0 and -8 volts. The path of operation of the tube is along the d-c load line. [See equation (21).] When $e_c = 0$, $e_b = 85$ volts, and when $e_c = -8$ volts, $e_b = 210$ volts. Since the output voltage is Δe_b,

$$\Delta E_{\text{out}} = 85 - 210 = -125 \quad \text{volts}$$

and since the corresponding change in grid voltage is

$$\Delta E_c = 0 - (-8) = 8 \quad \text{volts}$$

$$A = \frac{\Delta E_{\text{out}}}{\Delta E_c} = -\frac{125}{8} = -15.6$$

which compares favorably with the value of voltage gain previously determined.

In closing this brief introduction to vacuum tube circuit analysis, attention is called to the fact that the μE_s voltage generator of Fig. 22 can be readily transformed into an equivalent $g_m E_s$ current generator by the method suggested in Section 17 of Chapter VI. With either type of generator, the unilateral nature of the transmission which takes place *from* grid *to* plate must be recognized. When the tube is operated as indicated in Figs. 21 and 22, a signal voltage introduced into the grid circuit (the k to g path) manifests itself in the form of an amplified voltage at the plate of the tube, that is, across the load resistor R_L. An a-c voltage introduced in series with R_L will not, however, be transmitted backward through the tube to the grid terminal except for the incidental transmission which trickles through the plate-to-grid capacitance.

14. The Transistor.[4] In 1948, the Bell Telephone Laboratories demonstrated a new type of amplifying unit which contained neither a vacuum nor a cathode heater. This unit consists essentially of a piece of semi-conductor (like germanium) to which two point contacts or "cat's whiskers" are applied as shown in Fig. 23.

The point contacts (one called the *emitter* and the other the *collector*) are separated from one another by so small a distance (0.002 in.) that the boundary layer rectifying properties of the germanium at the collector are affected by the potential gradient established at the *surface area of interaction* by the emitter. The change produced in the molecular structure of the boundary layer in the immediate vicinity of the emitter by the input current results in a corresponding and amplified change in

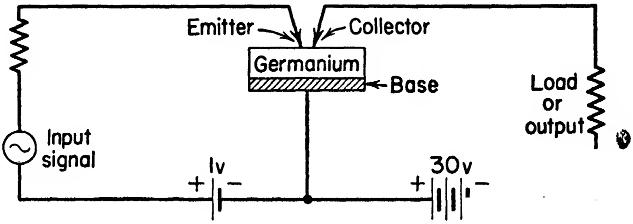


FIG. 23. Transistor amplifier.

the output current. Thus the output or load current can be controlled by the input current, and current amplifications of 100 to 1 have been obtained in experimental models.

The similarity in the amplifier circuits shown in Figs. 21 and 23 is at once apparent, and it is reasonable to expect that, in the future, the transistor will take its place beside the vacuum tube in engineering applications where the mechanism of current amplification is involved.

PROBLEMS

1. The I versus V characteristic of a particular thyrite disk is found experimentally to be

V, volts	I, ma	V, volts	I, ma	V, volts	I, ma
0	0	40	2.56	80	40.96
10	0.01	50	6.25	90	65.61
20	0.16	60	12.96	100	100.00
30	0.81	70	24.01		

Write the equation for I as a function of V with I given in milliamperes when V is expressed in volts.

2. If a 200-ohm fixed resistor is connected in series with the thyrite disk of Prob. 1 across a voltage of 110 volts, what current will flow in the series circuit?

[4] See *Electrical Engineering*, August 1948, page 740.

3. If the thyrite resistor of Prob. 1 is placed in series with the thyrite resistor of Fig. 6 (page 415) across a voltage 120 volts, what current will flow in the series circuit?

4. A 1000-ohm fixed resistor, the thyrite resistor of Prob. 1, and the thyrite resistor of Fig. 6, page 415, are connected in series across a voltage of 100 volts. What current will flow in the series circuit?

5. The resistance of a particular iron oxide thermal resistor is known to be 20,000 ohms at 30°C. Find the resistance at 200°C by means of equation (6), page 417, if $C = 171{,}280$, $b = 1306.5$, and $s = 2.8638$.

6. From the definition of the temperature coefficient of resistivity, α, it is plain that

$$\alpha_K = \frac{1}{R_K} \frac{dR}{dT}\bigg]_{T=T_K}$$

where K refers to some specified temperature. Assuming that the resistance of a thermal resistor follows the exponential law given in equation (7), page 418, show that

$$\alpha_K = -\frac{\beta}{T_K^2}$$

where T_K is expressed in degrees Kelvin.

7. Find the resistivity temperature coefficient of the uranium oxide resistor of Fig. 8, page 417, at 0°C if $\Delta(1/T)$ of Fig. 9 is 0.78×10^{-3} inverse temperature units as indicated in Fig. 9.

8. A particular thermal resistor has the following temperature-resistance relationship:

T°C	25	50	75	100
R ohms	284	99	40	19

(*a*) Evaluate β of equation (7), page 418, for this particular temperature range.
(*b*) Check the result obtained in (*a*) by assuming that the 25°C resistance is known (284 ohms) and calculating the 100°C resistance with the aid of equation (7).

9. The resistance of a particular oxide resistor can be represented reasonably accurately by equation (7), page 418, between 0°C and 100°C if β has the value of 2730° Kelvin. If the resistance at 0°C is known to be 10,000 ohms, find the temperature of the resistor when the resistance has dropped to 1000 ohms.

10. A particular negative-temperature-coefficient resistor has the following temperature-resistance relationship:

Temperature, °C	Resistance, ohms	Temperature, °C	Resistance, ohms
−20	350	+60	157
0	270	+80	143
+20	212	+100	130
+40	180		

This resistor (R_{NTC} of Fig. 24) is paralleled with a 200-ohm alloy resistor R_s which has a resistivity temperature coefficient which is negligibly small. In series with this parallel combination is a copper wound coil R_a, the 0°C resistance of which is 85 ohms.

(a) What is the resistance R_{xy} of Fig. 24 at −20°C and at 100°C?

(b) What is the approximate maximum deviation of R_{xy} in ohms from 200 ohms over the temperature range −20°C to +100°C?

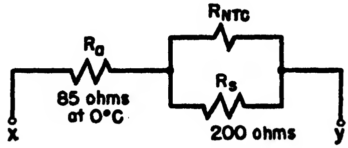


FIG. 24. See Prob. 10.

11. The I versus V characteristic of the diode shown in Fig. 18-b is

$$I = (2 \times 10^{-4})V^{3/2} \text{ amp} \quad (\text{for } +V \text{ only})$$

where V is expressed in volts. $e = 120 \sin (5000t)$ volts; $R_L = 100$ ohms. What is the time-averaged value of current in the series circuit over one complete cycle of the applied voltage e?

12. A type 6C5 tube is operated under the following conditions:

$$E_c = -2 \text{ volts} \quad E_{bo} = 130 \text{ volts} \quad R_L = 21{,}000 \text{ ohms}$$

Find the direct operating current I_{bo} and the plate supply voltage E_{bb}.

13. What are the values of g_m, r_p, and μ of a type 6C5 tube operated with a grid bias voltage of $E_c = -2$ volts and a plate current $I_{bo} = 8$ ma?

14. Find the effective value of the output voltage E_{out} in Fig. 21-a if the triode is a 6C5, $e_s = 2 \sin (500t)$ volts, and

$$E_c = -2 \text{ volts} \quad E_{bb} = 300 \text{ volts} \quad R_L = 21{,}000 \text{ ohms}$$

Neglect the capacitive reactance of the output condenser.

15. What is the gain of the amplifier described in Prob. 14?

16. Transform the μE_s voltage generator of Fig. 22 to an equivalent current generator, and find the output of this generator in milliamperes if $\mu = 15$, $r_p = 10{,}000$ ohms, and $E_s = 1$ volt effective.

Wire Table, Standard Annealed Copper

American Wire Gage (B & S). English Units

Gage No. A.W.G.	Diameter in Mils at 20° C.	Cross Section at 20° C.		Ohms per 1000 ft.* at 20° C. (= 68° F.)	Pounds per 1000 ft.	Feet per Pound	Feet per Ohm † at 20° C. (= 68° F.)	Ohms per Pound at 20° C. (= 68° F.)	Pounds per Ohm at 20° C. (= 68° F.)
		Circular mils	Square inches						
0000	460.0	211 600.	0.1662	0.049 01	640.5	1.561	20 400.	0.000 076 52	13 070.
000	409.6	167 800.	0.1318	0.061 80	507.9	1.968	16 180.	0.000 121 7	8219.
00	364.8	133 100.	0.1045	0.077 93	402.8	2.482	12 830.	0.000 193 5	5169.
0	324.9	105 500.	0.082 89	0.098 27	319.5	3.130	10 180.	0.000 307 6	3251.
1	289.3	83 690.	0.065 73	0.123 9	253.3	3.947	8 070.	0.000 489 1	2044.
2	257.6	66 370.	0 052 13	0.156 3	200.9	4.977	6 400.	0.000 777 8	1286.
3	229.4	52 640.	0.041 34	0.197 0	159.3	6.276	5 075.	0.001 237	808.6
4	204.3	41 740.	0.032 78	0.248 5	126.4	7.914	4 025.	0.001 966	508.5
5	181.9	33 100.	0.026 00	0.313 3	100.2	9.980	3 192.	0.003 127	319.8
6	162.0	26 250.	0.020 62	0.395 1	79.46	12.58	2 531.	0.004 972	201.1
7	144.3	20 820.	0.016 35	0.498 2	63.02	15.87	2 007.	0.007 905	126.5
8	128.5	16 510.	0.012 97	0.628 2	49.98	20.01	1 592.	0.012 57	79.55
9	114.4	13 090.	0.010 28	0.792 1	39.63	25.23	1 262.	0.019 99	50.03
10	101.9	10 380.	0.008 155	0.998 9	31.43	31.82	1 001.	0.031 78	31.47
11	90.74	8234.	0.006 467	1.260	24.92	40.12	794.0	0.050 53	19.79
12	80.81	6530.	0.005 129	1.588	19.77	50.59	629.6	0.080 35	12.45
13	71.96	5178.	0.004 067	2.003	15.68	63.80	499.3	0.1278	7.827
14	64.08	4107.	0.003 225	2.525	12.43	80.44	396.0	0.2032	4.922
15	57.07	3257.	0.002 558	3.184	9.858	101.4	314.0	0.3230	3.096
16	50.82	2583.	0.002 028	4.016	7.818	127.9	249.0	0.5136	1.947
17	45.26	2048.	0.001 609	5.064	6.200	161.3	197.5	0.8167	1.224
18	40.30	1624.	0.001 276	6.385	4.917	203.4	156.6	1.299	0.770 0
19	35.89	1288.	0.001 012	8.051	3.899	256.5	124.2	2.065	0.484 3
20	31.96	1022.	0.000 802 3	10.15	3.092	323.4	98.50	3.283	0.304 6
21	28.46	810.1	0.000 636 3	12.80	2.452	407.8	78.11	5.221	0.191 5
22	25.35	642.4	0.000 504 6	16.14	1.945	514.2	61.95	8.301	0.120 5
23	22.57	509.5	0.000 400 2	20.36	1.542	648.4	49.13	13.20	0.075 76
24	20.10	404.0	0.000 317 3	25.67	1.223	817.7	38.96	20.99	0.047 65
25	17.90	320.4	0.000 251 7	32.37	0.969 9	1 031.	30.90	33.37	0.029 97
26	15.94	254.1	0.000 199 6	40.81	0.769 2	1 300.	24.50	53.06	0.018 85
27	14.20	201.5	0.000 158 3	51.47	0.610 0	1 639.	19.43	84.37	0.011 85
28	12.64	159.8	0.000 125 5	64.90	0.483 7	2 067.	15.41	134.2	0.007 454
29	11.26	126.7	0.000 099 53	81.83	0.383 6	2 607.	12.22	213.3	0.004 688
30	10.03	100.5	0.000 078 94	103.2	0.304 2	3 287.	9.691	339.2	0.002 948
31	8.928	79.70	0.000 062 60	130.1	0.241 3	4 145.	7.685	539.3	0.001 854
32	7.950	63.21	0.000 049 64	164.1	0.191 3	5 227.	6.095	857.6	0.001 166
33	7.080	50.13	0.000 039 37	206.9	0.151 7	6 591.	4.833	1 364.	0.000 733 3
34	6.305	39.75	0.000 031 22	260.9	0.120 3	8 310.	3.833	2 168.	0.000 461 2
35	5.615	31.52	0.000 024 76	329.0	0.095 42	10 480.	3.040	3 448.	0.000 290 1
36	5.000	25.00	0.000 019 64	414.8	0.075 68	13 210.	2.411	5 482.	0.000 182 4
37	4.453	19.83	0.000 015 57	523.1	0.060 01	16 660.	1.912	8 717.	0.000 114 7
38	3.965	15.72	0.000 012 35	659.6	0.047 59	21 010.	1.516	13 860.	0.000 072 15
39	3.531	12.47	0.000 009 793	831.8	0.037 74	26 500.	1.202	22 040.	0.000 045 38
40	3.145	9.888	0.000 007 766	1 049.0	0.029 93	33 410.	0.953 4	35 040.	0.000 028 54

* Resistance at the stated temperatures of a wire whose length is 1000 ft. at 20° C.
† Length at 20° C. of a wire whose resistance is 1 ohm at the stated temperatures.

APPENDIX B

Chemical Elements Referred to in This Text

Element	Symbol	Atomic Number	Atomic Weight (1938)	Usual Valence	Nature
Aluminum	Al	13	26.98	3	metal
Argon	A	18	39.944	0	inert gas
Barium	Ba	56	137.36	2	metal
Beryllium	Be	4	9.02	2	metal
Boron	B	5	10.82	3	metalloid
Bromine	Br	35	79.916	1	liquid
Cadmium	Cd	48	112.41	2	metal
Calcium	Ca	20	40.08	2	metal
Carbon	C	6	12.01	2,4	metalloid
Cesium	Cs	55	132.91	1	metal
Chlorine	Cl	17	35.457	1	gas
Chromium	Cr	24	52.01	(2),3,6	metal
Cobalt	Co	27	58.94	2,(3)	metal
Copper	Cu	29	63.57	1,2	metal
Fluorine	F	9	19.00	1	most active gas
Gold	Au	79	197.2	1,3	metal
Helium	He	2	4.003	0	inert gas
Hydrogen	H	1	1.0081	1	lightest gas
Iodine	I	53	126.92	1	metalloid
Iron	Fe	26	55.84	2,3	metal
Krypton	Kr	36	83.7	0	inert gas
Lead	Pb	82	207.21	2,(4)	metal
Lithium	Li	3	6.940	1	metal
Magnesium	Mg	12	24.32	2	metal
Manganese	Mn	25	54.93	2,(4,6),7	metal
Mercury	Hg	80	200.61	1,2	metal
Neon	Ne	10	20.183	0	inert gas
Nickel	Ni	28	58.69	2,3	metal
Nitrogen	N	7	14.008	3,5	gas
Oxygen	O	8	16.000	2	gas
Phosphorus	P	15	31.02	3,5	metalloid
Platinum	Pt	78	195.23	(2),4	metal
Potassium	K	19	39.096	1	metal
Rubidium	Rb	37	85.48	1	metal
Scandium	Sc	21	45.10	3	metal
Silicon	Si	14	28.06	4	metalloid
Silver	Ag	47	107.880	1	metal
Sodium	Na	11	22.997	1	metal
Strontium	Sr	38	87.63	2	metal
Sulphur	S	16	32.06	2,4,6	metalloid
Tin	Sn	50	118.70	2,4	metal
Titanium	Ti	22	47.90	3,4	metal
Tungsten	W	74	183.92	6	metal
Uranium	U	92	238.07	4,6	metal
Vanadium	V	23	50.95	3,5	metal
Xenon	Xe	54	131.3	0	inert gas
Zinc	Zn	30	65.38	2	metal

APPENDIX C

Energy Levels of Orbital Electrons

Atomic Number	Chemical Element	Nuclear Charge	Different Energy Levels (Orbital Electrons in Different Levels) *K*	*L*	*M*	*N*	*O*
1	Hydrogen	1	1				
2	Helium	2	2				
3	Lithium	3	2	1			
4	Beryllium	4	2	2			
5	Boron	5	2	3			
6	Carbon	6	2	4			
7	Nitrogen	7	2	5			
8	Oxygen	8	2	6			
9	Fluorine	9	2	7			
10	Neon	10	2	8			
11	Sodium	11	2	8	1		
12	Magnesium	12	2	8	2		
13	Aluminum	13	2	8	3		
14	Silicon	14	2	8	4		
15	Phosphorus	15	2	8	5		
16	Sulphur	16	2	8	6		
17	Chlorine	17	2	8	7		
18	Argon	18	2	8	8		
19	Potassium	19	2	8	8	1	
20	Calcium	20	2	8	8	2	
21	Scandium	21	2	8	(9)	(2)	
22	Titanium	22	2	8	(10)	(2)	
23	Vanadium	23	2	8	(11)	(2)	
24	Chromium	24	2	8	(12)	(2)	
25	Manganese	25	2	8	(13)	(2)	
26	Iron	26	2	8	(14)	(2)	
27	Cobalt	27	2	8	(15)	(2)	
28	Nickel	28	2	8	(16)	(2)	
29	Copper	29	2	8	18	1	
30	Zinc	30	2	8	18	2	
31	Gallium	31	2	8	18	3	
32	Germanium	32	2	8	18	4	
33	Arsenic	33	2	8	18	5	
34	Selenium	34	2	8	18	6	
35	Bromine	35	2	8	18	7	
36	Krypton	36	2	8	18	8	
37	Rubidium	37	2	8	18	8	1
38	Strontium	38	2	8	18	8	2
....	....	....					
47	Silver	47	2	8	18	18	1

Note: Numbers in parentheses are uncertain. The energy levels given above are what we shall call the primary levels of ideal atoms. Experimental evidence points to the fact that the primary levels are often broken up into subdivisions and under these conditions the orbital electrons in a given primary level will not have exactly the same energy. This can give rise to several types of atoms which differ only slightly from the ideal atom but this matter will not be considered in this text.

APPENDIX D

Conversion Chart for Units of Energy

Multiply Number of → by / To Obtain	Electron Volts	Ergs	Joules (watt-sec or newton-meters)	Foot-pounds (ft-lb)	Gram-calories (g-cal)	British thermal units (Btu)	Kilowatt-hours (kwhr)
Electron volts	1	6.25×10^{11}	6.25×10^{18}	8.475×10^{18}	26.16×10^{18}	6594×10^{18}	2.25×10^{25}
Ergs	1.6×10^{-12}	1	10^{7}	1.356×10^{7}	4.186×10^{7}	1055×10^{7}	3.6×10^{13}
Joules	1.6×10^{-19}	10^{-7}	1	1.356	4.186	1055	3.6×10^{6}
Foot-pounds	1.18×10^{-19}	7.367×10^{-8}	0.7376	1	3.087	778	2.655×10^{6}
Gram-calories	3.82×10^{-20}	2.388×10^{-8}	0.2388	0.3239	1	252	8.60×10^{5}
British thermal units	1.516×10^{-22}	9.48×10^{-11}	9.48×10^{-4}	1.285×10^{-3}	3.97×10^{-3}	1	3413
Kilowatt-hours	4.44×10^{-26}	2.778×10^{-14}	2.778×10^{-7}	3.766×10^{-7}	1.163×10^{-6}	2.93×10^{-4}	1

(Other conversion factors are given on page 1–140 of Eshbach's *Handbook of Engineering Fundamentals.*)

APPENDIX E

Useful Numerical Values

Magnitude of the charge of an electron	1.6×10^{-19} coulomb
Mass of an electron	9.1×10^{-31} kg
Permittivity of free space (rationalized mks units) .	$\dfrac{1}{36\pi \times 10^{9}} = 8.842 \times 10^{-12}$
Permeability of free space (rationalized mks units) .	$4\pi \times 10^{-7} = 1.257 \times 10^{-6}$
Number of circular mils per square inch	1.273×10^{6}
Number of circular mils per square centimeter . .	1.973×10^{5}
Resistivity temperature coefficient of copper . . .	0.00427 at 0°C
Resistivity temperature coefficient of copper . . .	0.00393 at 20°C

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